An Interesting Proof of Pauli Identity

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Theorem 1. *Pauli Identity:*

$$\sum_{a=1,2,3} \sigma^a_{\alpha\beta}\sigma^a_{\gamma\delta} = 2\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta} \tag{1}$$

with σ *the pauli matrices.*

Proof. The great orthoganality theorm can be used to prove this:

$$\sum_{a=1,2,3} \sigma^a_{\alpha\beta}\sigma^a_{\gamma\delta} = 2\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta} \tag{2}$$

where σ are pauli matrices. Note that 2-by-2 Pauli matrices belongs to irreducible representation of SU(2), so we can apply the theorem directly to Pauli group:

$$G_P = \{\pm I, \pm\sigma^x, \pm\sigma^y, \pm\sigma^z, \pm iI, \pm i\sigma^x, \pm i\sigma^y, \pm i\sigma^z\} \tag{3}$$

Now we apply the GOT:

$$\sum_{g\in G_P} D^\dagger(g)_{\alpha\beta} D(g)_{\gamma\delta} = 8\delta_{\alpha\delta}\,\delta_{\gamma\beta} \tag{4}$$

Note that $\pm$ and i has no effect on the L.H.S. due to the mutiplication with the complex conjuate, the summation $\sum_{g\in G_p}$ can be divided into 4 identical sums, of which we are interested the subset $P = \{I, \sigma^x, \sigma^y, \sigma^z\}$. This gives:

$$\sum_{g\in P} D^\dagger(g)_{\alpha\beta} D(g)_{\gamma\delta} = 2\delta_{\alpha\delta}\delta_{\gamma\beta} \tag{5}$$

we can separate out the identity I out of the sum, that is, whcih gives us $\delta_{\alpha\beta}\delta_{\gamma\delta}$ on the L.H.S., and write the representation explicit by pauli matrices σ:

$$\delta_{\alpha\beta}\delta_{\gamma\delta} + \sum_a \sigma^a_{\alpha\beta}\sigma^a_{\gamma\delta} = 2\delta_{\alpha\delta}\delta_{\gamma\beta} \tag{6}$$

rearrange and we have the desired form:

$$\sum_{a=1,2,3} \sigma^a_{\alpha\beta}\sigma^a_{\gamma\delta} = 2\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta} \tag{7}$$

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