Burnside's Lemma

Shi Feng

January 20, 2022

1 Preliminaries

Definition 1.1. Orbits: Let G be a finite group and $m \in M$ is a point in space M. We define the **orbit of m under G** as the set given by

$$G \cdot m = \{am \mid a \in G\}$$

Definition 1.2. Isotropy Group (Stabilizer): The isotropic (sub)group of m in G, also termed a **stabilizer**, is the set of elements in G that leave m invariant. It is written as G_m.

Given any point $m \in M$ we can consider the subset G_m of G consisting of those $a \in G$ which satisfy $am = m$, i.e. the point m remains invariant under these operations. It is simple to see that such a subset forms a subgroup of G because

1. $\mathbb{I}m = m$, G_m has an identity element.
2. if $am = m$, then $a^{-1}m = m \Rightarrow a^{-1} \in G_m$. So all elements in G_m have inverse.
3. if $am = m$, $bm = m$, then $(ab)m = m$. That is $\forall a, b \in G$ and $a \neq b$, we have $ab \in G$. Hence the closure condition is met.

We call such a subgroup G_m of G the *isotropy group of m*.

Definition 1.3. Transformer: Given a group G that acts on set M, the transformer of two set elements $m, n \in M$, denoted by $trans(m, n)$ or S_{mn}, is defined as:

$$trans(m, n) = S_{mn} = \{a \in G | am = n\}$$

Note that $S_{mm} = G_m$.

$\forall a \in G,\ m \in M$, there is a bijection f between the orbit $G \cdot m$ and the set of left cosets $L_m = \{aG_m | \forall a \in G\}$ i.e. the bijection

$$f : G \cdot m \to L_m$$

given that $f : am \mapsto aG_m$.

Proof. (1) f is well-defined: Let $y \in G \cdot m$ being a point in orbit, we need to show that different representatives of y is mapped to the same left coset. Let the two representatives be $y = a_1m = a_2m$ with $a_1, a_2 \in G$, we immediately have

$$a_2^{-1}a_1m = a_2^{-1}a_2m = m \Rightarrow a_2^{-1}a_1 \in G_m$$

Therefore

$$a_2^{-1}a_1G_m = G_m \Rightarrow a_1G_m = a_2G_m$$

so we have $y = a_1m = a_2m \Rightarrow a_1G_m = a_2G_m$, hence f is well-defined.

(2) f is surjective, which is self-explanatory by the definition $f : am \mapsto aG_m$.

(3) f is injective: We already know f is well-defined and surjective, so we only need to show that a coset is the image of the same element in orbit. If $aG_m = a'G_m$, then $\exists h \in G_m,\ a = a'h$. Then $am = (a'h)m = a'm$. QED □

Theorem 1.1 (Orbit-Stabilizer Theorem)**.** *Let G be a finite group and $m \in M$ is a point in space M. Let $|G|$ denote the cardinality of G. Then*

$$|G| = |G \cdot m||G_m|$$

Proof. According to Lagrangian theorem we can partition the group G by isotropy subgroup G_m into G/G_m, which is exactly the set of left cosets of isotropy group of m. That is

$$G = \bigcup\{G_m, a_1G_m, a_2G_m, \ldots\},\ \ a_iG_m \in G/G_m$$

From Lemma.1 we know that there is a bijection from these left cosets to orbit of m under G, therefore in the curly bracket there are total $|G \cdot m|$ of these cosets. Then it's readily to see $|G| = |G \cdot m||G_m|$ must hold. □

2 Burnside's Lemma

Lemma 2.1.

$$\sum_{a\in G} |M^a| = \sum_{m\in M} |G_m|$$

Proof. Let $Z \subset G \times M$, defined by

$$Z = \{(b, m)|bm = m, b \in G, m \in M\}$$

Define two functions θ and τ acting on Z by:

$$\rho(b, m) = m,\ \ \sigma(b, m) = b$$

which gives two fibers ρ^{-1}, σ^{-1} over M and G. Specifically, Z is fibered over M by ρ^{-1}, the fiber over a point $\rho^{-1}(m)$ being its isotropy group G_m; Z is also fibered over G by σ^{-1}, the fiber over a group element $\sigma^{-1}(a)$ being its fixed points M^a. That is

$$Z \cong \bigcup_{m\in M} G_m \times \{m\} = \bigcup_{a\in G} \{a\} \times M^a$$

This gives two ways to count $|Z|$. Using the fiber over G and the fiver over M respectively:

$$|Z| = \sum_{a\in G} |M^a|,\ \ |Z| = \sum_{m\in M} |G_m| \ \Rightarrow \sum_{a\in G} |M^a| = \sum_{m\in M} |G_m|$$

Hence we've shown the relationship between the summation over fixed points and that over isotropy group elements. □

Corollary 2.1.1. *Let elements of the quotient M/G (viz. set of orbits) labeled by $O_1, \dots, O_r$, then*

$$\sum_{m\in M} |G_m| = \sum_{O_i} |G|$$

A simple intuitive example: Suppose G is a symmetry group of M, that is $G = G_m, \forall m \in M$, and each orbit has single element $G \cdot m = \{m\}$. Then it's readily to see the corollary holds.

Proof.

$$\sum_{m\in M} |G_m| = \sum_{O_i} \sum_{m\in O_i} |G_m| = \sum_{O_i} |G \cdot m||G_m| = \sum_{O_i} |G|$$

where in the 2nd step we have used the fact that isotropy groups of elements that belong to the same orbit has the same cardinality, which is readily to see from $G_{am} = aG_m a^{-1}$ whereby am being an arbitrary member of $O(m)$. □

Lemma 2.2. ***Burnside's Lemma:*** *Let M/G be the set of orbits of M, then, Burnside's lemma states that*

$$|M/G| = \frac{1}{|G|} \sum_{a\in G} |M^a|$$

where M^a is the subset of M whose elements are invariant under $a \in G$, that is, M^a is the set of fixed points $FP(a)$.

Proof. Note that every element in a orbit $G \cdot m$ contributes $1/|G \cdot m|$ to the total sum of orbits i.e. its sum over all elements in an orbit gives $\sum_{m'\in G\cdot m} 1/|G \cdot m| = 1$. Therefore

$$|M/G| = \sum_{G\cdot m} \sum_{m'\in G\cdot m} \frac{1}{|G \cdot m|} = \sum_{m\in G} \frac{1}{|G \cdot m|}$$

Then the orbit-stabilizer theorem tells that $1/|G \cdot m| = |G_m|/|G|$, hence by Lemma.2.1:

$$|M/G| = \frac{1}{|G|} \sum_{m\in G} |G_m| = \frac{1}{|G|} \sum_{a\in G} |M^a|$$

□