Understanding Boring Hamiltonians

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1 Spins in orthogonal fields

In quantum mechanics, Ising model is the simpliest non-trivial toy model, whose local energy reads $\sigma_i^z\sigma_{i+1}^z + g\sigma_i^x$. However, we can readily write down an even simpler but boring Hamiltonian:

$$H = \sum_i \sigma_i^z + g\sum_i \sigma_i^x \tag{1}$$

where the n.n. coupling is absent. Yet, like Ising model, there is competition between z-polarized state and x-polarized state nonetheless and the two contributions to Hamiltonian do not commute with each other. However, this apparent competition turns out to be **trivial** under some rotation.

This can be proved by constructing a rotation about y-axis by some angle θ, such that the Hamiltonian becomes a single pauli matrix afterwards. The rotation about y-axis by θ is given by the unitary operator:

$$R = \exp(-i\theta S_y) = \exp\left(-i\frac{\theta}{2}\sigma_y\right) = \cos\left(\frac{\theta}{2}\right) - i\sigma^y \sin\left(\frac{\theta}{2}\right) \tag{2}$$

so that for a single site, σ^z becomes

$$\begin{aligned} R^\dagger\sigma^z R &= \left[\cos\left(\frac{\theta}{2}\right) + i\sigma^y \sin\left(\frac{\theta}{2}\right)\right]\sigma^z\left[\cos\left(\frac{\theta}{2}\right) - i\sigma^y \sin\left(\frac{\theta}{2}\right)\right] \\ &= \sigma^z\cos\theta - \sigma^x\sin\theta \end{aligned} \tag{3}$$

and the second term in Eq.1 becomes:

$$\begin{aligned} R^\dagger(g\sigma^x)R &= g\left[\cos\left(\frac{\theta}{2}\right) + i\sigma^y \sin\left(\frac{\theta}{2}\right)\right]\sigma^x\left[\cos\left(\frac{\theta}{2}\right) - i\sigma^y \sin\left(\frac{\theta}{2}\right)\right] \\ &= \sigma^x g\cos\theta + \sigma^z g\sin\theta \end{aligned} \tag{4}$$

so that the onsite Hamiltonian density is

$$h_i = (\cos\theta + g\sin\theta)\sigma^z + (g\cos\theta - \sin\theta)\sigma^x \tag{5}$$

with $H = \sum_i h_i$. Now let us define θ:

$$\theta = \tan^{-1} g \tag{6}$$

such that the second term in Eq.5 becomes

$$g\cos\theta - \sin\theta = \cos\theta(g - \tan\theta) = \cos(g - g) = 0 \tag{7}$$

and the first term in Eq.5:

$$\cos\theta + g\sin\theta = \cos\theta(1+g\tan\theta) = \cos\theta(1+g^2) = \sqrt{1+g^2} \tag{8}$$

where we used $\cos\theta = 1/\sqrt{1+g^2}$. Therefore, by a global rotation $\prod_i R_i$ the Hamiltonian is essentially a trivial one:

$$H = \sqrt{1+g^2}\sum_i \sigma_i^z \tag{9}$$

Therefore we won't see any phase transition or singularity as we tune g even if σ_x and σ_z doesn't commute: the continuous symmetry is always present and will never break into discrete ones.

2 Hopping Fermions

The most boring Fermionic Hamiltonian one can write down is

$$H = tc_1^\dagger c_2 + tc_2^\dagger c_1 \tag{10}$$

For convenience we write it in the matrix form:

$$H = (c_1\ c_2)\begin{pmatrix}0 & t\\ t & 0\end{pmatrix}\begin{pmatrix}c_1^\dagger\\ c_2^\dagger\end{pmatrix} \tag{11}$$

In order to rotate $t\sigma_x$ to a diagonal matrix, i.e. into σ_z, we again apply a rotation about y axis by $R = \exp\left(-i\frac{\theta}{2}\sigma_y\right)$.

$$R^\dagger\sigma^x R = \sigma^x\cos\theta + \sigma^z\sin\theta \tag{12}$$

setting $\theta = \frac{\pi}{2}$ gives $R = \frac{\sqrt{2}}{2} - i\sigma^y\frac{\sqrt{2}}{2}$ and $R^\dagger\sigma^x R = \sigma^z$. So the resulting Hamiltonian is

$$H = t\hat{\psi}\sigma^z\hat{\psi}^\dagger = t\hat{\psi}_1^\dagger\hat{\psi}_1 - t\hat{\psi}_2^\dagger\hat{\psi}_2 \tag{13}$$

where the normal mode is given by

$$\hat{\psi}^\dagger = \begin{pmatrix}\hat{\psi}_1^\dagger\\ \hat{\psi}_2^\dagger\end{pmatrix} = R^\dagger\begin{pmatrix}c_1^\dagger\\ c_2^\dagger\end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}c_1^\dagger\\ c_2^\dagger\end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix}c_1^\dagger + c_2^\dagger\\ c_2^\dagger - c_1^\dagger\end{pmatrix} \tag{14}$$

so the eigen states of the Hamiltonian besides $|0\rangle$ are

$$|\psi_1\rangle = \frac{\sqrt{2}}{2}(c_1^\dagger + c_2^\dagger)|0\rangle\,,\ \ |\psi_2\rangle = -\frac{\sqrt{2}}{2}(c_1^\dagger - c_2^\dagger)|0\rangle \tag{15}$$

At half filling, $|\psi_1\rangle$ is the excited state with energy t and $|\psi_2\rangle = |\psi_g\rangle$ is the ground state with energy $-t$. By the same token we can write down eigen states for spinful hopping particles whose Hamiltonian is

$$H = t\sum_\sigma c_{1,\sigma}^\dagger c_{2,\sigma} + c_{2,\sigma}^\dagger c_{1,\sigma} \tag{16}$$

where $\sigma = \pm$ denotes $\uparrow$ and $\downarrow$. At one-particle filling (which is not half-filling for spinful particle! half-filling for spinful two-site system has two particles!), the ground state energy is two-fold degenerate:

$$|\psi_{g,\pm}\rangle = -\frac{\sqrt{2}}{2}(c_{1,\pm}^\dagger - c_{2,\pm}^\dagger)|0\rangle\,,\ \ E_{g,\pm} = -t \tag{17}$$

whose magnetization per site is

$$\langle\psi_{g,\pm}|S_i^z|\psi_{g,\pm}\rangle = \pm\frac{1}{4} \tag{18}$$

Therefore for different cat states

$$|\psi_g(\alpha)\rangle = \alpha\,|\psi_{g,+}\rangle + \sqrt{1-\alpha^2}\,|\psi_{g,-}\rangle \tag{19}$$

the magnetization can be different. Numeraically, to break this cat-state symmetry one has to add a small pinning potential.

At half-filling (two-particle filling), the Hamiltonian in the diagonal basis reads

$$H = t\hat{\psi}^\dagger_{1,\uparrow}\hat{\psi}_{1,\uparrow} - t\hat{\psi}^\dagger_{2,\uparrow}\hat{\psi}_{2,\uparrow} + t\hat{\psi}^\dagger_{1,\downarrow}\hat{\psi}_{1,\downarrow} - t\hat{\psi}^\dagger_{2,\downarrow}\hat{\psi}_{2,\downarrow} \tag{20}$$

The ground state then has to fill $\psi_{2,\uparrow}$ and $\psi_{2,\downarrow}$, both with energy $-t$, hence

$$|\psi_g\rangle = \hat{\psi}^\dagger_{2,\uparrow}\hat{\psi}^\dagger_{2,\downarrow}\,|0\rangle \tag{21}$$

By Eq.15 dressed with spin, we have

$$|\psi_g\rangle = \frac{1}{2}(c^\dagger_{1,\uparrow}c^\dagger_{1,\downarrow} + c^\dagger_{2,\uparrow}c^\dagger_{2,\downarrow} - c^\dagger_{1,\uparrow}c^\dagger_{2,\downarrow} - c^\dagger_{2,\uparrow}c^\dagger_{1,\downarrow})\,|0\rangle \tag{22}$$