On Black Body Radiation

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1 Planck Distribution Function

Let the energy of s-th photon mode denoted by $\epsilon_s = s\hbar\omega$. The probability that the system is in the state of s of energy ϵ_s is given by the Boltzmann factor:

$$P(s) = \exp(-s\hbar\omega/k_BT) \tag{1.1}$$

then the thermal average value of s is

$$\langle s\rangle = \sum_{s=0}^{\infty} sP(s) = \frac{1}{Z}\sum_{s=0}^{\infty} s\exp(-s\hbar\omega/k_BT) \tag{1.2}$$

with $y \equiv \hbar\omega/k_BT$, the summation can be written as

$$\begin{aligned}\sum_{s=0}^{\infty} s\exp(-sy) &= -\frac{d}{dy}\sum_{s=0}^{\infty}\exp(-sy)\\ &= -\frac{d}{dy}\left(\frac{1}{1-\exp(-y)}\right) = \frac{\exp(-y)}{[1-\exp(-y)]^2} = \frac{\exp(-\hbar\omega/k_BT)}{[1-\exp(-\hbar\omega/k_BT)]^2}\end{aligned} \tag{1.3}$$

and the partition function can be evaluated as

$$Z = \frac{1}{1-\exp(-\hbar\omega/k_BT)} \tag{1.4}$$

so we have

$$\langle s\rangle = \frac{\exp(-\hbar\omega/k_BT)}{1-\exp(-\hbar\omega/k_BT)} \tag{1.5}$$

or equivalently

$$\langle s\rangle = \frac{1}{\exp(\hbar\omega/k_BT)-1} \tag{1.6}$$

This is called the Plank distribution function which essentially converges to the Bose-Einstein distribution. Note that $\langle s\rangle$ is dependent of frequency ω. Here $\langle s(\omega)\rangle$ means the thermal average of the number of photons in the mode of frequency ω.

2 Plank Law and Stefan-Boltzmann Law

According to previous section, the thermal average energy of mode s is

$$\langle\epsilon_s\rangle = \langle s\rangle\,\hbar\omega = \frac{\hbar\omega}{\exp(\hbar\omega/k_BT)-1} \tag{2.1}$$

The high temperature limit $k_BT \gg \hbar\omega$ is dubbed classical limit, where $\exp(\hbar\omega/k_BT)$ may be approximated as

$$\exp(\hbar\omega/k_BT) \approx 1 + \hbar\omega/k_BT + \cdots \tag{2.2}$$

thus in the classical limit the thermal average energy of the mode is

$$\langle \epsilon_s \rangle \approx k_BT \tag{2.3}$$

which is only a temperature of T and no longer depends on ω. This is consistent with the equal partition theorem of dof = 1.

Generically, we need to sum over all modes with their respective energy ϵ_n such that the total energy is to be given by $U = \sum_n \langle \epsilon_n \rangle$. To do this, we first need to define all available modes and the allowed frequencies ω_n (Ref. K&K, P93 [111 of 495]). The result from electrodynamics gives us

$$\omega_b = n\pi c/L \tag{2.4}$$

then the total energy is

$$U = \sum_n \langle \epsilon_n \rangle = \sum_n \frac{\hbar\omega_n}{\exp(\hbar\omega_n/k_BT) - 1} \tag{2.5}$$

where $n \equiv \sqrt{n_x^2 + n_y^2 + n_z^2}$ with n_i integers. We replace the sum over n_x, n_y, n_z by $dn_x dn_y dn_z$ in space. The summation is then written in integral as

$$\sum_n f(n) \approx \iiint_0^\infty f(n) dn_x dn_y dn_z = \frac{1}{8}\int_0^\infty 4\pi n^2 f(n) dn \tag{2.6}$$

where 1/8 is due to $n_i \geq 0$ thus only the positive octant of the space is involved, and the last step we assumed the isotropy of photon energy density. We now multiply the sum or integral by a factor of2 because there are two independent polarizations of the electromagnetic field (two independent sets of cavity modes). Thus, using $\omega_n = \pi cn/L$ we have

$$\begin{aligned} U &= \pi \int_0^\infty dn\, n^2 \frac{\hbar\omega_n}{\exp(\hbar\omega_n/k_BT) - 1} \\ &= \frac{\pi^2\hbar c}{L}\int_0^\infty dn\, n^3 \frac{1}{\exp(\hbar nc\pi/Lk_BT) - 1} \end{aligned} \tag{2.7}$$

to evaluate this, we set $x \equiv \pi\hbar cn/Lk_BT$, thus $n = (Lk_BT/\pi\hbar c)x$ and $dn = (Lk_BT/\pi\hbar c)dx$, then the integral becomes

$$U = \frac{\pi^2\hbar c}{L}\left(\frac{k_BTL}{\pi\hbar c}\right)^4 \int_0^\infty dx \frac{x^3}{\exp(x) - 1} \tag{2.8}$$

the definite integral evaluates to $\pi^4/15$. Then, the energy per unit volume is found to be

$$u \equiv \frac{U}{V} = \frac{\pi^2 k_B^4}{15\hbar^3 c^3} T^4 \tag{2.9}$$

with $V = L^3$. The result that the radiant energy density is proportional to the fourth power of the temperature is known as the **Stefan-Boltzmann law** of radiation.

In applications we also would like to know the energy density spectrum resolved in frequency ω. To do this we simply invert the relation between ω_n and n into

$$n = \omega_n L/\pi c \quad \Rightarrow \quad dn = \frac{L}{\pi c} d\omega \tag{2.10}$$

then the integral for U becomes

$$U = \pi\hbar \left(\frac{L}{\pi c}\right)^3 \int_0^\infty d\omega \, \frac{\omega^3}{\exp(\hbar\omega/k_B T) - 1} \tag{2.11}$$

so the energy per volume is

$$\frac{U}{V} = \frac{\hbar}{\pi^2 c^3} \int_0^\infty d\omega \frac{\omega^3}{\exp(\hbar\omega/k_B T) - 1} \tag{2.12}$$

hence the spectral density is

$$u(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{\exp(\hbar\omega/k_B T) - 1} \tag{2.13}$$

3 Black body

The measurement of high temperatures depends on the flux of radiant energy from a small hole in the wall of a cavity maintained at the temperature of interest. Such a hole is said to radiate as a black body, which means that the radiation emission is characteristic of a thermal equilibrium distribution.

Define the energy flux J_U as the rate of energy per unit area, that is, the the amount of energy that passes through a unit area per unit time. Then we can straightforwardly write:

$$J_U = c\frac{U(T)}{V} \times (\text{geometrical factor}) \tag{3.1}$$