Toric Code

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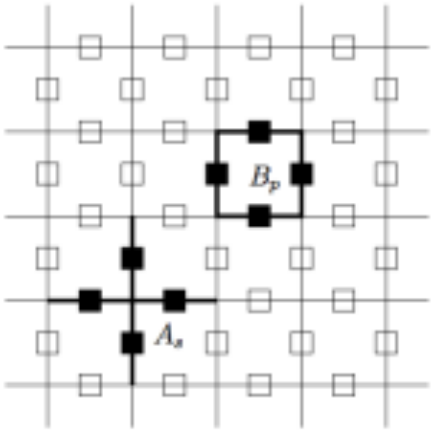
Introduction

The Toric code Hamiltonian:

$$H_{TC} = -J_1 \sum_c A_s - J_2 \sum_p B_p$$

where $A_s = \prod_s \sigma_i^x, \;\; B_p = \prod_p \sigma_i^z$



Ground state construnction

Hamiltonian is made of purely commuting terms

$$[A_s, A_{s'}] = 0$$
$$[B_p, B_{p'}] = 0$$
$$[A_s, B_p] = 0$$

so that both plaquette and star operators commute with Hamiltonian:

$$[A_s, H] = [B_p, H] = 0$$

A_s and B_p can be simultaneously diagonalized. Assuming $J > 0$, the ground state is when all $B_p = 1$ and $A_s = 1$

The pictorial solution

Work in σ_z basis. The classical configuration: $s_l = \pm 1$.
The ground state is some superposition of vortex-free configurations. We must have:

$$B_p \left|\psi_0\right\rangle = \left|\psi_0\right\rangle \quad \Rightarrow \quad \left|\psi_0\right\rangle = \sum_{v.f.} c_s \left|s\right\rangle$$

A_s is a good quantum number, which evaluates to $+1$ at g.s.

$$A_s \left|\psi_0\right\rangle = \left|\psi_0\right\rangle$$

This condition holds true if and only if all the c_s are equal for each orbit of the A_s

Gauge point of view

View A_s as a gauge transformation operator. Physical states must satisfy:

$$A_s |\Psi_0\rangle = |\Psi_0\rangle$$

Start with the trivial $|\Psi_0\rangle = \bigotimes_l |s_l = 1\rangle$, which is not gauge invarient since apparently A_s will flip spins on 4 links thus $A_s |\Psi_0\rangle \neq |\Psi_0\rangle$. Such a local gauge transformation can be fixed by redefining our wavefunction:

$$|\Psi\rangle = |\Psi_0\rangle + A_s |\Psi_0\rangle$$

such that

$$A_s |\Psi\rangle = A_s |\Psi_0\rangle + A_s^2 |\Psi_0\rangle = A_s |\Psi_0\rangle + |\Psi_0\rangle$$

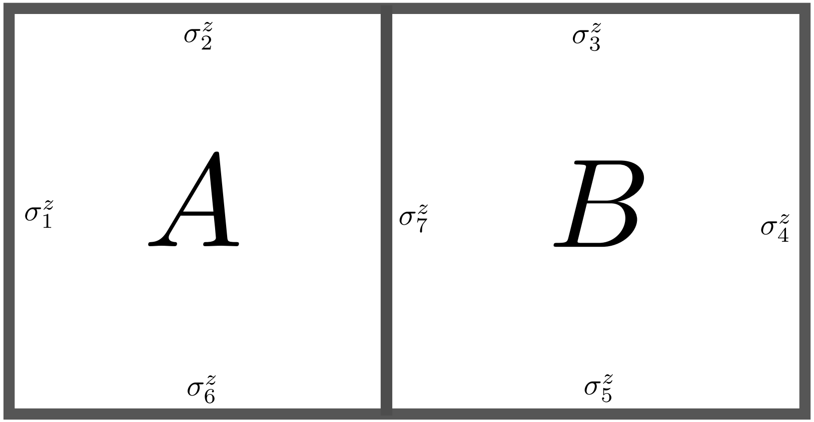
Therefore the ground state is:

$$|\Psi\rangle \propto \prod_s (1 + A_s) |\Psi_0\rangle$$

Essentially, we're superposing all gauge-equivalent wavefunction into one gauge-equivalent class.

Contractable loops

The prodect of σ^z eigenvalues of the links of **any closed loop** in the **Gound state** is always 1: $\prod_{r \in \{\text{closed loop}\}} \sigma^z_r = 1$



$$\sigma^z_1\sigma^z_2\sigma^z_3\sigma^z_4\sigma^z_5\sigma^z_6 = (\sigma^z_1\sigma^z_2\sigma^z_7\sigma^z_6)_B(\sigma^z_3\sigma^z_4\sigma^z_5\sigma^z_7)_A$$
$$= B_A\, B_B = 1$$

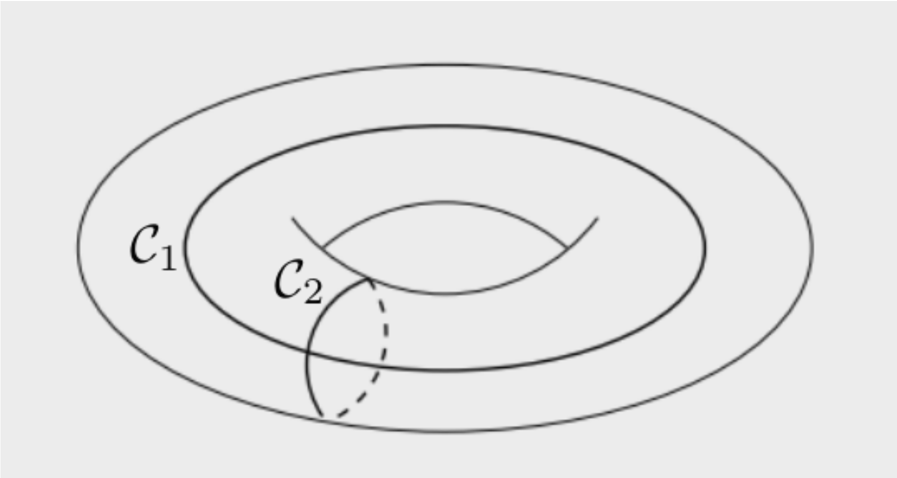
where we have used $\sigma^z_7\sigma^z_7 = 1$

Degeneracy - non-contractable loops on $\mathbb{T}^2$

Define Wilson-loop operator:

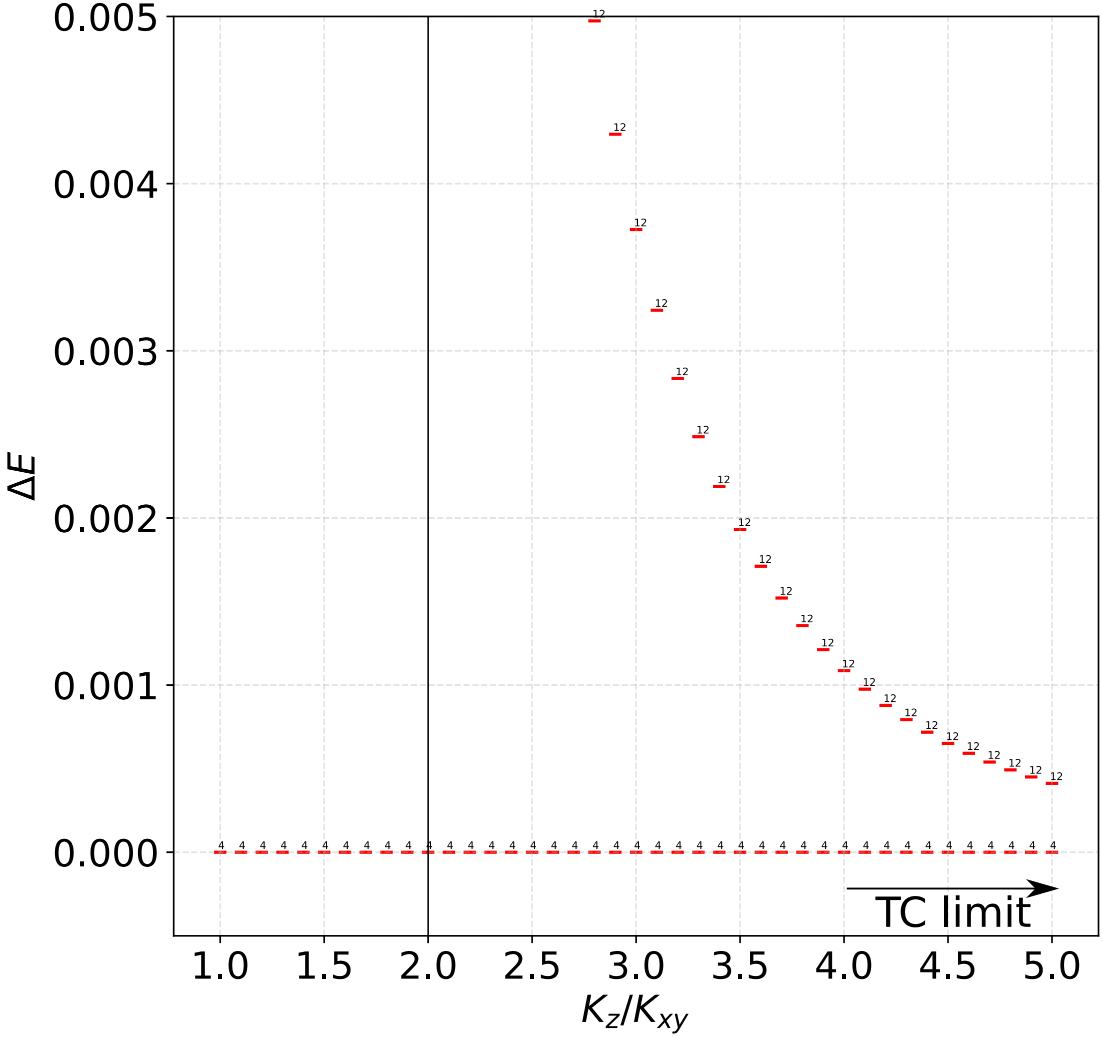
$$W_{\mathcal{C}}(s) = \prod_{l \in \mathcal{C}} s_l, \quad \mathcal{C} = \mathcal{C}_1 \ \ or \ \ \mathcal{C}_2$$

This forms "superselection" sectors, i.e. $W_{\mathcal{C}}$ is unaffected by A_s.



$W_{\mathcal{C}_{1,2}} = \pm 1 \quad \Rightarrow \quad$ 4-fold degenerate ground state.

TC limit - Numerical results

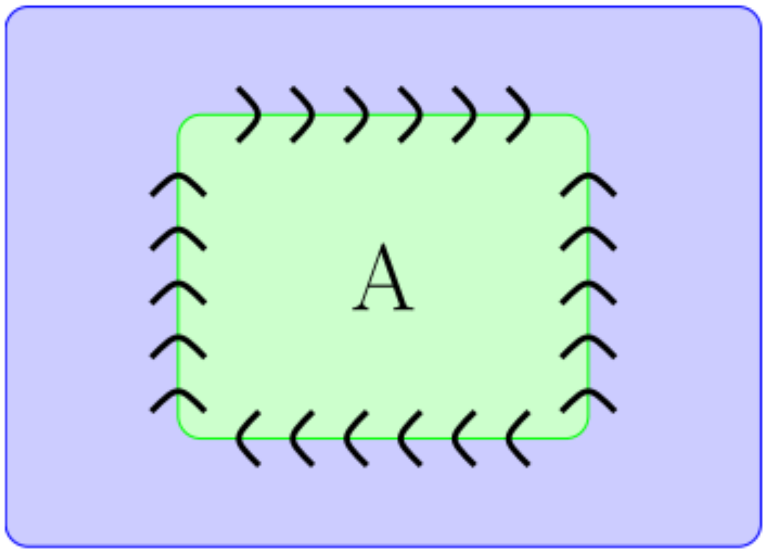


Entanglement Entropy

Scaling of entanglement in 2D Gapped system:

$$S_A \sim \alpha L$$

– the "Area law". L being perimeter of closed loop

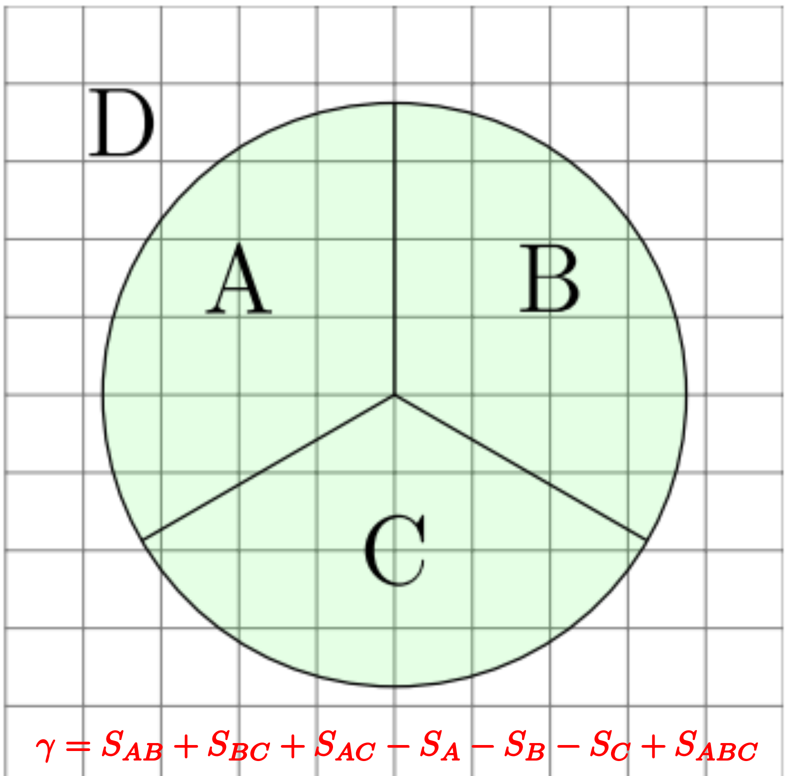


Entanglement entropy: topologically ordered states

Additional term γ: *Topological entanglement entropy*

$$S_A \sim \alpha L - \gamma$$

$\gamma \neq 0$ indicates long-range entanglement structure that originates from the topological nature of the system.



$$\gamma = S_{AB} + S_{BC} + S_{AC} - S_A - S_B - S_C + S_{ABC}$$

γ_{TC}=log 2

The entanglement entropy in a rectangular region

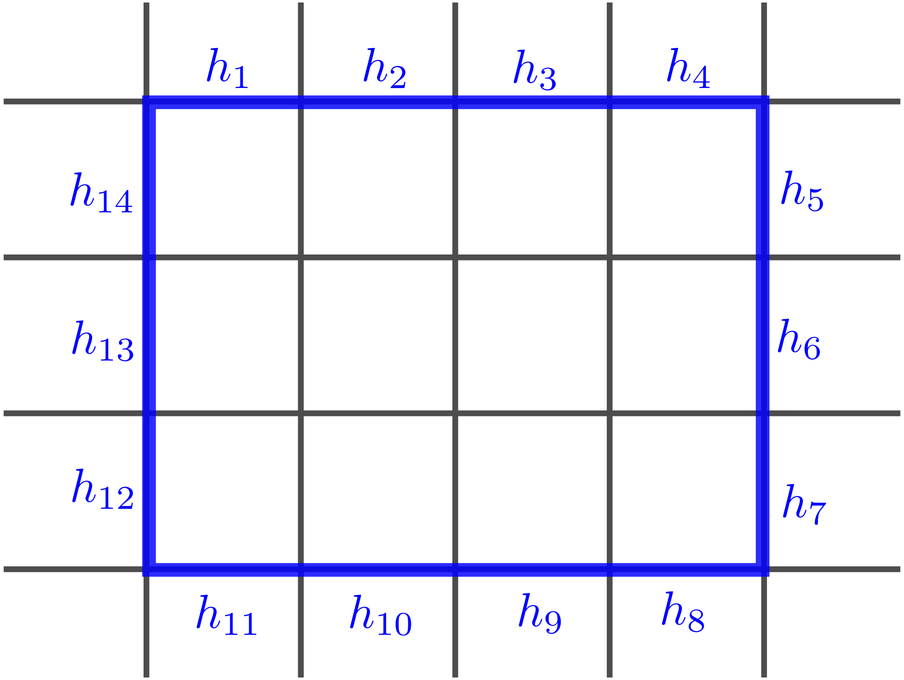


Figure 1: Degrees of freedoms live on links, the boundary of the rectangular area is labeled by h_i.

The ground state is $\{h\}-$ *dependent*:

$$\left|\psi_{\{h_i\}}\right\rangle = |h_1, h_2, \ldots, h_n\rangle \otimes \left|\psi_{\{h_i\}}, in\right\rangle \otimes \left|\psi_{\{h_i\}}, out\right\rangle .$$

(This is a product state of 3 sectors in the Schmidht basis)

Then the full ground state is:

$$|\psi\rangle \propto \sum_{\{h_i\}} \left|\psi_{\{h_i\}}\right\rangle = \sum_{\{h_i\}} |h_1, \ldots, h_n\rangle \otimes \left|\psi_{\{h_i\}}, in\right\rangle \otimes \left|\psi_{\{h_i\}}, out\right\rangle .$$

We apply this result to the rectangular partition of lattice:

$$\prod_{r \in \{\mathrm{C.L.}\}} \sigma_r^z \left| h_1, \ldots, h_n \right\rangle = 1 \;\; or \;\; h_1 \times h_2 \times \ldots \times h_n = 1.$$

Therefore, the boundary sector $\left| h_1, \ldots, h_n \right\rangle$ has 2^{n-1} independent configurations.

The normalized ground state is then:

$$\left|\psi\right\rangle = \frac{1}{2^{(n-1)/2}} \sum_{\{h_i\}} \left| h_1, \ldots h_n \right\rangle \left| \psi_{\{h_i\}}, in \right\rangle \left| \psi_{\{h_i\}}, out \right\rangle .$$

The density matrix is then:

$$\rho = |\psi\rangle \langle\psi| = \sum_{\{h_i\}} \sum_{\{h'_i\}} \left|\psi_{\{h_i\}}\right\rangle \left\langle\psi_{\{h'_i\}}\right|$$

$$= \frac{1}{2^{n-1}} \sum_{\{h_i\}} \sum_{\{h'_i\}} \Big(|h_1 \ldots h_n\rangle \; |\psi_{\{h_i\}}, in\rangle \; |\psi_{\{h_i\}}, out\rangle \Big) \Big(h.c.' \Big)$$

Trace out *out* sector:

$$\rho_{in} = \frac{1}{2^{n-1}} |h_1 \ldots h_n\rangle \; |\psi_{\{h_i\}}, in\rangle \; \langle h_1 \ldots h_n| \; \langle\psi_{\{h_i\}}, in|$$

which is exactly $\mathbb{I}_{2^{n-1} \times 2^{n-1}}$

$$\rho_{in} = \frac{1}{2^{n-1}} \left| h_1 \dots h_n \right\rangle \left| \psi_{\{h_i\}}, in \right\rangle \left\langle h_1 \dots h_n \right| \left\langle \psi_{\{h_i\}}, in \right| \equiv \mathbb{I}_{2^{n-1} \times 2^{n-1}}.$$

Therefore the entanglement entropy is:

$$S_{EE} = -tr[\rho_{in} \log \rho_{in}] = (n-1)\log 2 = \boxed{n \log 2 - \log 2}$$

where the first term $n \log 2$ The same result can be dereived from

PK construction:

$$S_{topo} = S_A + S_B + S_C - S_{AB} - S_{BC} - S_{AC} + S_{ABC} = -\log 2.$$

Vac.

The Hamiltonian is:

$$H_{TC} = -J_1 \sum_c A_s - J_2 \sum_p B_p$$

where $A_s = \prod_s \sigma_i^x, \;\; B_p = \prod_p \sigma_i^z$. Ground state is vortex-free:

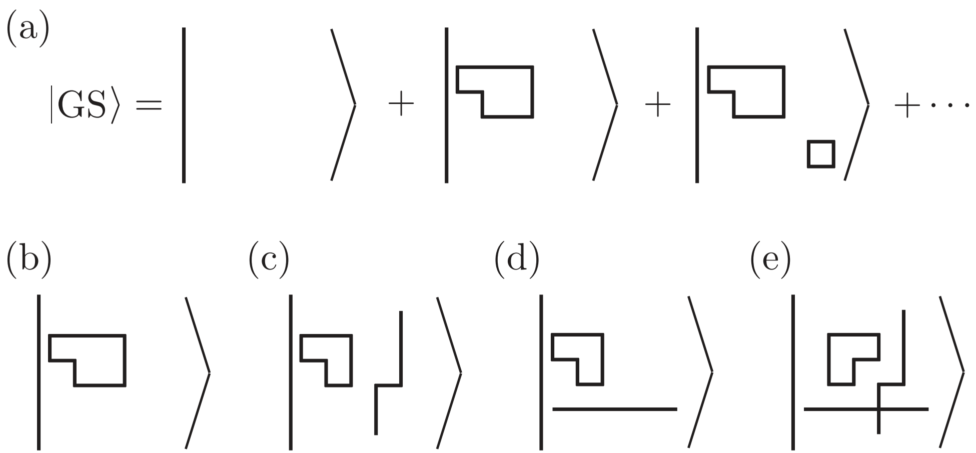


Figure 2: Illustration of G.S. by classical configuration

Charge Excitation

Define electric-path operator:

$$W_{\mathbb{C}}^{(e)}(s_1, s_2) = \prod_{l \in \mathbb{C}} \tau_l^z$$

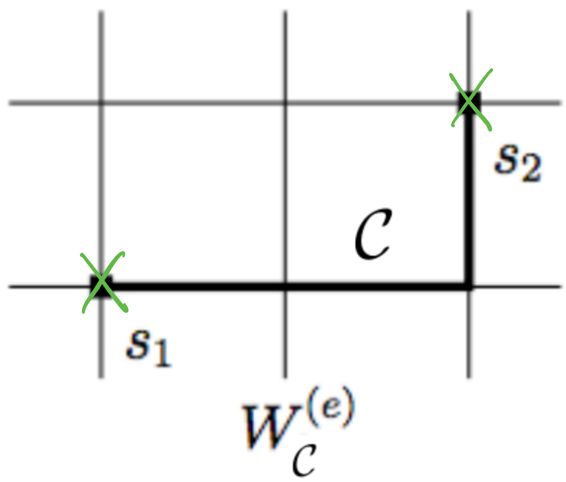
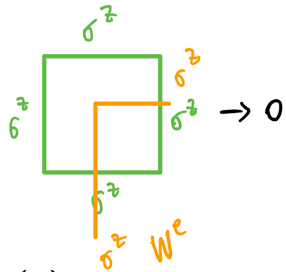
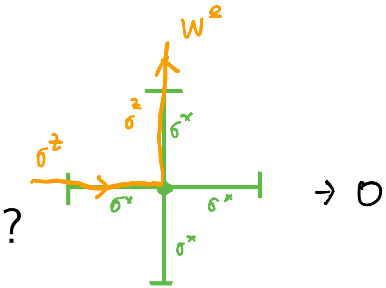


Figure 3:

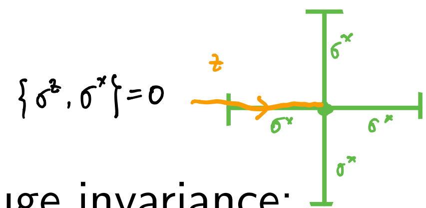


$$[W_{\mathbb{C}}^{(e)}, B_p] = 0, \quad [W_{\mathbb{C}}^{(e)}, A_s] = ?$$



$W_{\mathbb{C}}^{(e)}$ commutes with most but not all star operators.
At the end points of electric path $\mathbb{C}$, which we label A_{s_1} and A_{s_2}:

$$\{W_{\mathbb{C}}^{(e)}, A_{s_{1/2}}\} = 0.$$



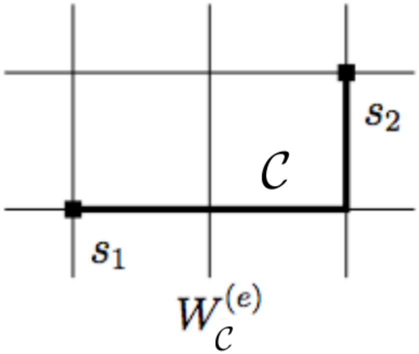
Let it act on the ground state wavefunction, by gauge invariance:

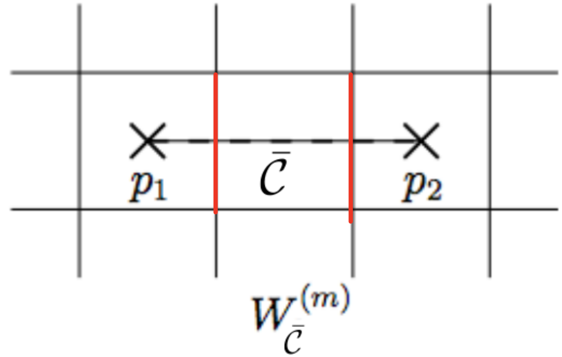
$$W_{\mathbb{C}}^{(e)}(s_1, s_2)\,|\Psi_0\rangle = -A_{s_{1/2}} W_{\mathbb{C}}^{(e)}(s_1, s_2)\,|\Psi_0\rangle$$

This flip the sign of local energy of A_s. So

$\boxed{|\Psi_{s_1,s_2}\rangle \equiv W_{\mathbb{C}}^{(e)}(s_1, s_2)\,|\Psi_0\rangle}$ is an eigenstate with energy $4J_e$.

Magnetic Vortices





Define an magnetic path operator $W_{\mathbb{C}}^{(m)}(p_1, p_2)$:

$$W_{\mathbb{C}}^{(m)}(p_1, p_2) = \prod_{l \in \mathbb{C}} \tau_l^x$$

where p_1 and p_2 are labels of plaquettes, and path $\mathbb{C}$ is path on dual lattice (centers of the meshgrid). $\boxed{\text{l} \in \mathcal{C} \text{ if they cut cross.}}$

$$[W_{\mathbb{C}}^{(m)}, A_s] = 0, \quad [W_{\mathbb{C}}^{(m)}, B_p] =?$$

All but two plaquette operators B_{p_1} and B_{p_2} at the ends of path $\mathbb{C}$ commute with $W_{\mathbb{C}}^{(m)}$.

$$\{W_{\mathbb{C}}^{(m)}(p_1, p_2), B_{p_{1/2}}\} = 0.$$

Similary to the charge excitation:

$$B_{p_1} |\Psi_{p_1,p_2}\rangle = - |\Psi_{p_1,p_2}\rangle$$

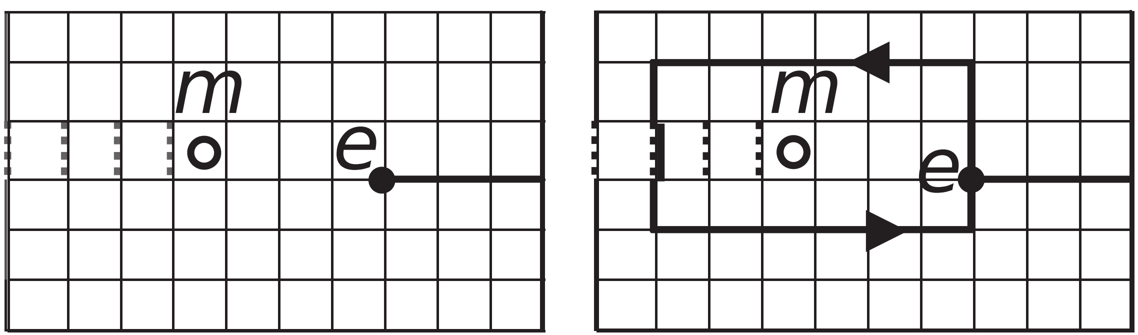
'magnetic fluxes' (m-particles) at the plaquettes p_1 and p_2 , each costs $2J_m$ to create.

Mutual Statistics

Take a charge e around a vortex m. Let $|\xi\rangle$ be a state contatining a magnetic vortex at p_1. Let $\mathbb{C}$ be a closed loop around p_1, then the braiding operation is defined as:

$$\left(\prod_{l \in \mathbb{C}} \tau_l^z\right) |\xi\rangle = \left(\prod_{p \in \mathcal{A}_{\mathbb{C}}} B_p\right) |\xi\rangle$$

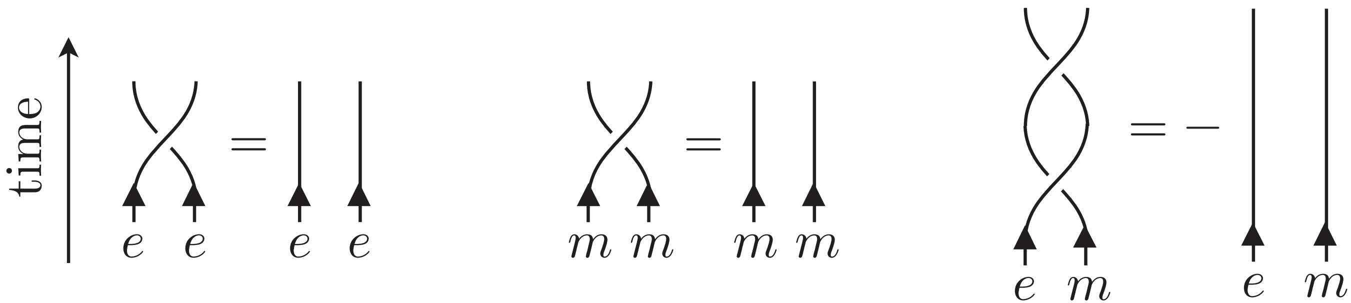
R.H.S is the lattice-version of Stokes' theorem



We have shown that $m-$particle flips sign of B_{p_1}, so that:

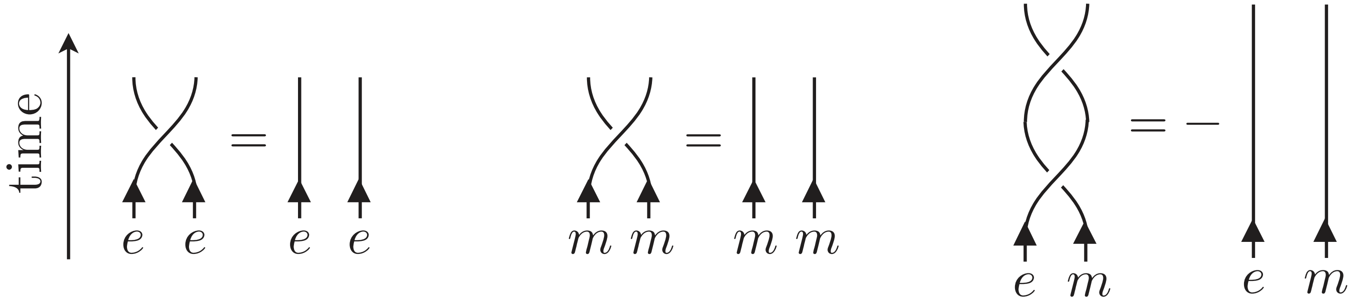
$$B_{p_1}|\xi\rangle = -|\xi\rangle \quad \Rightarrow \quad \left(\prod_{p\in\mathcal{A}_{\mathbb{C}}} B_p\right)|\xi\rangle = -|\xi\rangle$$

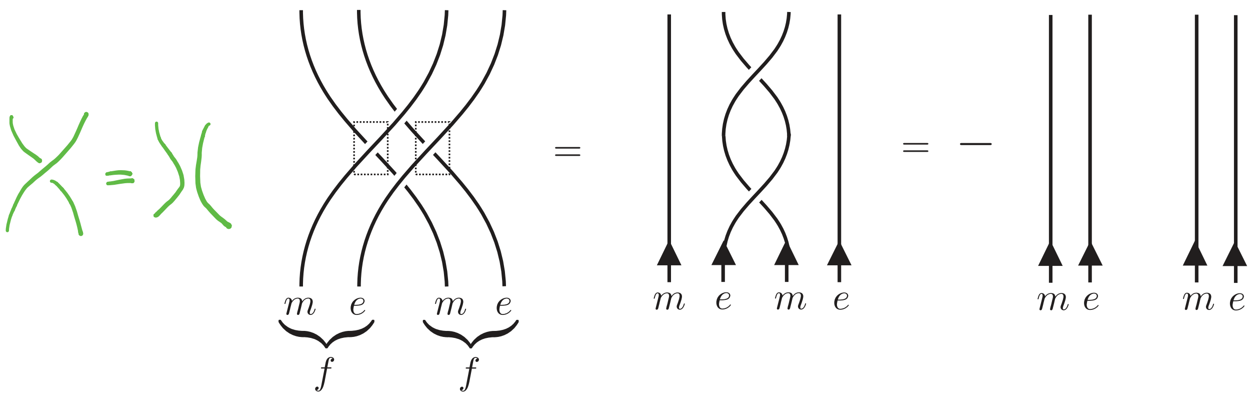
upon braiding e around m, wavefunction changes by $|\xi\rangle \to -|\xi\rangle$, i.e. we pick up a phase of π. This gives the fusion rule:



*exchange twice is topologically equivalent to braiding around.

Fusion Rule





$$e \times e = 1, \quad m \times m = 1, \quad e \times m = f$$