Detect Topological Entropy by Local Measurements

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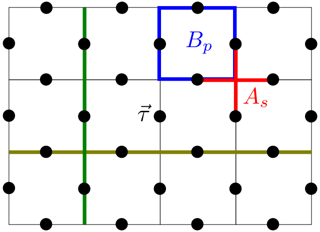
Hamiltonian of a $\mathbb{Z}_2$ Gauge Theory

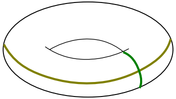
Kitaev's Toric code Hamiltonian:

$$\mathcal{H} = -\sum_s A_s - \sum_p B_p$$



$$B_p = \prod_{\square} \tau_i^z$$

$$A_s = \prod_{+} \tau_i^x$$



Topological Signatures:

1. Four-fold ground state degeneracy
2. Wilson Loops that defines four superselection sectors:

$$(W_x, W_y) = (\pm 1, \pm 1)$$

3. Emergent anyon statistics
4. Topological entanglement entropy γ

$$S_{vN}(L) = \alpha L - \gamma + O(L^{-1})$$

Is it possible to extract γ locally?

Operational Approach

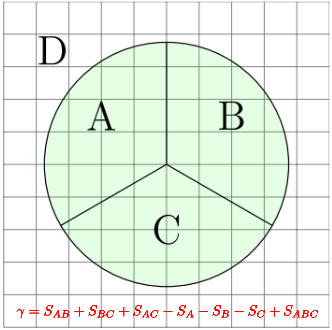
S_{vN} can be quantified operationally by comparing it to the entanglement of a reference system, usually taken to be a set of N Bell pairs

$$N_{\text{Bell}} = -\operatorname{Tr}(\rho \log \rho)$$

But this cannot detect γ

Kitaev-Preskill Approach and Mutual Information

Kitaev-Preskill Construction:

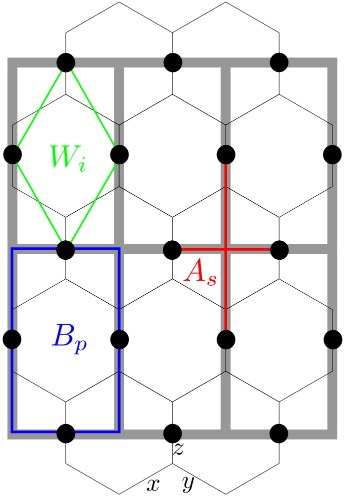


Third Order Mutual Information:

$$I_3(A, B, C) = -\sum_{a,b,c} p(a, b, c) \frac{p(a, b, c)p(a)p(b)p(c)}{p(a, b)p(b, c)p(a, c)}$$
$$= S(a) + S(b) + S(c)$$
$$- S(a, b) - S(a, c) - S(b, c) + S(a, b, c)$$
$$\simeq \gamma$$

Non-dyadic information
shared between three subsystems.

Embed $\mathbb{Z}_2$ Gauge Field into KSL



$$\mathcal{H} = K_x \sum_{\text{x-bonds}} \sigma_i^x \sigma_j^x + K_y \sum_{\text{y-bonds}} \sigma_i^y \sigma_j^y + K_z \sum_{\text{z-bonds}} \sigma_i^z \sigma_j^z$$

1. QSL with emergent $\mathbb{Z}_2$ gauge theory
2. With matter **Majoranas** on the $\mathbb{Z}_2$ field
3. $K_z/K \gg 2 \Rightarrow$ TC

$$\mathcal{H} \to J_{\text{TC}} \sum_i W_i \Rightarrow J_{\text{TC}} \sum_s A_s + \sum_p B_p$$

with $\tau^z = (\sigma_a^z - \sigma_b^z)/2$.

Local Reduced Density Matrix in KSL

One-point RDM:

$$\rho = \mathrm{diag}(a, 1-a) + b\sigma^x + c\sigma^y, \ \ \langle\sigma^\alpha\rangle = \mathrm{Tr}(\rho\sigma^\alpha)$$

$$a = \frac{1+\langle\sigma^z\rangle}{2} = \frac{1}{2}$$

$$b = \frac{\langle\sigma^x\rangle}{2} = 0; \ \ c = \frac{\langle\sigma^y\rangle}{2} = 0$$

Hence

$$\rho = \mathrm{diag}(\frac{1}{2}, \frac{1}{2}) \ \Rightarrow \ S_{vN} = \log 2$$

Two-point (p, q) RDM:

$$\rho = \frac{1}{4}\sum_{ij} \left\langle \sigma_p^i \sigma_q^j \right\rangle \sigma_p^i \sigma_q^j, \quad i, j \in \{0, 1, 2, 3\}$$

Correlation can be decomposed into gauge $|\mathcal{G}\rangle$ and matter sector $|M_{\mathcal{G}}\rangle$; only the latter contribute

$$\left\langle \sigma_p^\alpha \sigma_{p+\beta}^\alpha \right\rangle = \langle i c_p c_{p+\alpha}\rangle\, \delta_{\alpha,\beta}$$

$$\langle i c_p c_{p+z}\rangle = \frac{\sqrt{3}}{16\pi^2}\int_{\mathrm{BZ}} \frac{\epsilon_k}{\sqrt{\epsilon_k^2 + \Delta_k^2}} d^2\vec{k}$$

Local Reduced Density Matrix in KSL

$\left\langle \sigma_p^x \sigma_{p+x}^x \right\rangle \equiv 4A$; $\left\langle \sigma_p^y \sigma_{p+y}^y \right\rangle \equiv -4B$; $\left\langle \sigma_p^z \sigma_{p+z}^z \right\rangle \equiv 1 - 4C$:

$$\rho_x = \frac{1}{4}\mathbb{I}_4 + \frac{A}{4}\mathbb{J}_4$$

$$\rho_y = \frac{4}{4}\mathbb{I}_4 + \text{anti-diag}(B, -B, -B, B)$$

$$\rho_z = \text{diag}(\frac{1}{2} - C, C, C, \frac{1}{2} - C)$$

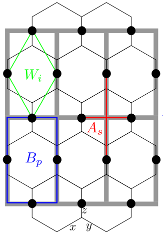
All of them has two-fold degeneracy due to TR-symmetry and conserved $\mathbb{Z}_2$ fluxes.

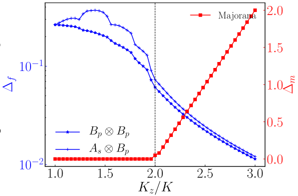
The von-Neumann entropies are:

$$S_{vN}(\alpha) = -2\sum_{\pm} \lambda_{\pm}^{\alpha} \log(\lambda_{\pm}^{\alpha})$$

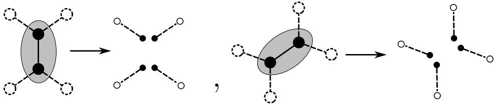
$$\lambda_{\pm}^{\alpha} = \frac{1}{4}(1 \pm \left\langle \sigma_p^{\alpha} \sigma_{p+\alpha}^{\alpha} \right\rangle)$$

Local Construction for TEE





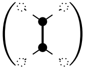
Decompose 2-site subsystem into formal Bell paris:

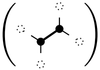


Observation:

1. Majorana are very gapped out at large K_z
2. Gauge charges A_s and B_p are conserved
3. A_s and B_p do not interact

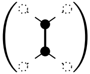
$$S_{vN}(L) \simeq \alpha L - \gamma$$

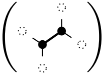
S_{vN}  $= 2S^{y}_{vN} + 2S^{x}_{vN} - \gamma$

S_{vN}  $= 2S^{y}_{vN} + 2S^{z}_{vN} - \gamma$

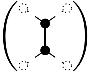
Local Construction for TEE

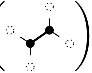
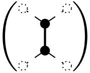
From decomposed S_{vN}

S_{vN} () $= 2S^{y}_{vN} + 2S^{x}_{vN} - \gamma$

S_{vN} () $= 2S^{y}_{vN} + 2S^{z}_{vN} - \gamma$

solve for S^{α}_{vN} of formal α-type Bell paris:

$S^{x}_{vN} = \frac{1}{4}\Big[S_{vN}$ () $+ \gamma \Big] = S^{y}_{vN}$

$S^{z}_{vN} = \frac{1}{2} S_{vN}$ () $- \frac{1}{4} S_{vN}$ () $+ \frac{\gamma}{4}$

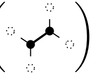
Consider $\mathcal{A} = \mathcal{S} \cup \mathcal{E}$ with n_x x bonds, n_y y bonds and n_z z bonds:

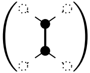
$$S_{vN}(\mathcal{S}) = n_{\alpha}(\mathcal{S}) S^{\alpha}_{vN} - \gamma$$

Its differential is:

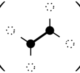
$$\Delta S_{vN} \equiv \Delta n_{\alpha} S^{\alpha}_{vN}$$

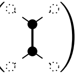
This gives:

$\gamma = \frac{1}{\sum_{\alpha} \Delta n_{\alpha}} \Big[4\Delta S_{vN} - 2\Delta n_z S_{vN}$ ()

$- (\Delta n_x + \Delta n_y - \Delta n_z) S_{vN}$ () $\Big]$

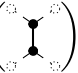
Extract $\gamma = \log 2$ in TC limit

$$\gamma = \frac{1}{\sum_\alpha \Delta n_\alpha}\Big[4\Delta S_{vN} - 2\Delta n_z S_{vN}\Big($$  $$\Big)$$

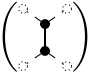
$$- (\Delta n_x + \Delta n_y - \Delta n_z) S_{vN}\Big($$  $$\Big)\Big]$$

$\langle \sigma_p^\alpha \sigma_{p+\alpha}^\alpha \rangle \propto \langle i c_p c_{p+\alpha} \rangle \to S_{vN}$ of 2-site dimers

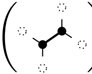
Choose ΔS_{vN} to be one and two site system:

$$\Delta S_{vN} = S_{vN}(\text{1-site}) - S_{vN}\Big($$  $$\Big)$$

From local RDM we know:

$$S_{vN}(\text{1-site}) = \log 2, \quad S_{vN}\Big($$  $$\Big) = \log 2$$

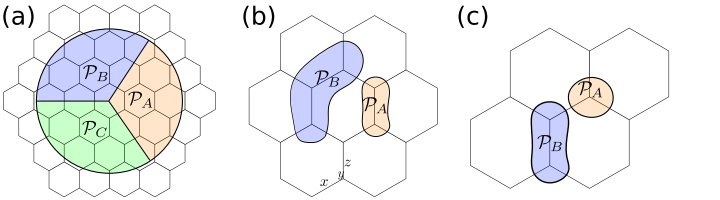
$$\Rightarrow \Delta S_{vN} = 0$$

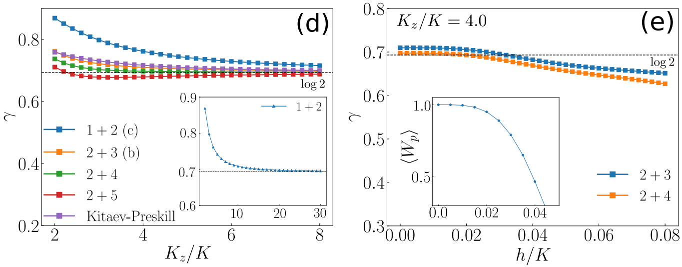
Plugging in $S_{vN}\Big($  $\Big) = 2\log 2$ and Δn_α:

$$\boxed{\gamma(\text{TC}) = \log 2}$$

Retrieved TEE by local measurement!

Away from TC limit





Bipartite Entropy