TRANSVERSE-FIELD ISING MODEL

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1 Perturbative Hamiltonian

Consider the TFIM Hamiltonian:

$$H_{TFIM} = -J\Big[\sum_i \sigma_i^z \sigma_{i+1}^z + g\sum_i \sigma^x\Big] \tag{1.1}$$

which has Z_2 symmetry defined by the operation:

$$U = \exp\left(i\pi \sum_j \frac{\sigma^x}{2}\right) = i^N \prod_j \sigma_j^x \sim \prod_j \sigma_j^x \tag{1.2}$$

This is readily apparent by anti-commutation $\{\sigma^x, \sigma^z\} = 0$.

Now assume we don't have the coupling term, so the ground state is a trivial polarized state:

$$|0\rangle = |\rightarrow\rightarrow\rightarrow \dots \rightarrow\rangle \tag{1.3}$$

There are a huge number of first excitated states:

$$|1\rangle = |\leftarrow\rightarrow\rightarrow \dots \rightarrow\rangle\,,\ |2\rangle = |\rightarrow\leftarrow\rightarrow \dots \rightarrow\rangle\,,\ |i\rangle = |\rightarrow\rightarrow \dots \leftarrow_i \dots \rightarrow\rangle \tag{1.4}$$

whose energy is $\Delta = 2gJ$ above g.s. E_0. They are solitons since there's no well-defined momentum or dispersion relation.

If we add the J terms perturbatively. To the leading order of perturbation we have:

$$H|i\rangle = -J[|i+1\rangle + |i-1\rangle] + (E_0 + 2gJ)|i\rangle \tag{1.5}$$

To show this we calculate $|\delta i\rangle$ by 1st order:

while the energy remains $E = E_0 + 2gJ$.

the exitations will then be able to tunnel to its nearby neig hbors' position, and they gains a well-defined dispersion.

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2 Mean Field Solution[1]

The Hamiltonian is written as:

$$H = -J \sum_{\langle ij \rangle} S_i^z S_j^z - h \sum_i S_i^x \tag{2.1}$$

Ignoring fluctuation:

$$S_i^z S_j^z = S_i^z \left\langle S_j^z \right\rangle + S_j^z \left\langle S_i^z \right\rangle - \left\langle S_i^z \right\rangle \left\langle S_j^z \right\rangle \tag{2.2}$$

due to translational symmetry:

$$\left\langle S_i^z \right\rangle = \left\langle S_j^z \right\rangle \equiv \left\langle S^z \right\rangle \tag{2.3}$$

so we rewrite the coupling term as:

$$S_i^z S_j^z = \left\langle S^z \right\rangle (S_i^z + S_j^z) - \left\langle S^z \right\rangle^2 \tag{2.4}$$

Leave off the constant $\left\langle S^z \right\rangle^2$, and apply $\sum_{\langle ij \rangle} = p/2 \sum_i$:

$$\begin{aligned} H &= -J \left\langle S^z \right\rangle \sum_{\langle ij \rangle} 2S_i^z - h \sum_i S_i^x \\ &= -\frac{pJ \left\langle S^z \right\rangle}{2} \sum_i \sigma_i^z - \frac{h}{2} \sum_i \sigma_i^x \end{aligned} \tag{2.5}$$

It's readily to see that the eigenvalue is:

$$\lambda = \pm \frac{1}{2} \sqrt{p^2 J^2 \left\langle S^z \right\rangle^2 + h^2} \tag{2.6}$$

The self-consistency equation is then:

$$\left\langle S^z \right\rangle = \frac{\mathrm{Tr}\left[S^z e^{-\beta H}\right]}{\mathrm{Tr}\, e^{-\beta H}} \tag{2.7}$$

In the diagonal basis, the denominator evaluates to:

$$Z = \mathrm{Tr}\left[e^{-\beta H}\right] = e^{\beta\lambda} + e^{-\beta\lambda} = \cosh(\beta\lambda) \tag{2.8}$$

The numerator is:

$$\begin{aligned} \mathrm{Tr}\left[S^z e^{-\beta H}\right] &= \frac{1}{N\beta} \frac{\partial \log Z}{\partial J'} = \frac{1}{2} \frac{pJ \left\langle S^z \right\rangle}{\sqrt{p^2 J^2 \left\langle S^z \right\rangle^2 + h^2}} (e^{\beta\lambda} - e^{-\beta\lambda}) \\ &= \frac{1}{2} \frac{pJ \left\langle S^z \right\rangle}{\sqrt{p^2 J^2 \left\langle S^z \right\rangle^2 + h^2}} \sinh(\beta\lambda) \end{aligned} \tag{2.9}$$

where we have defined $J' \equiv pJ \left\langle S^z \right\rangle / 2$. Therefore the average magnetization is:

$$\left\langle S^z \right\rangle = \frac{pJN \left\langle S^z \right\rangle}{2\sqrt{p^2 J^2 \left\langle S^z \right\rangle^2 + h^2}} \tanh\left[\frac{\beta}{2} \sqrt{p^2 J^2 \left\langle S^z \right\rangle^2 + h^2}\right] \tag{2.10}$$

At zero-temperature, $\tanh = 1$, so that:

$$\left\langle S^z \right\rangle_{T \to 0} = \frac{N}{2} \frac{pJ \left\langle S^z \right\rangle}{\sqrt{p^2 J^2 \left\langle S^z \right\rangle^2 + h^2}} \tag{2.11}$$

The magnetization $m \equiv \left\langle S^z \right\rangle$ vanishes at the critical value $(h/J)_c = p$, and obey scaling $m \propto |g|^{1/2}$ where $|g| = |(h/J) - (h/J)_c|$.

3 Majorana Fermions

To be consistent with reference *Whence QFT*, we write TFIM as

$$H = -J \sum_j (\sigma_j^z \sigma_{j+1}^z + g\sigma_j^x) \tag{3.1}$$

First we define our *Jordan-Wigner* transformation, that is, the order parameter we are to use is obtained by "attaching a spin to a domain wall":

$$\begin{aligned} \chi_j &\equiv \sigma_j^z \tau_{j+1/2}^z = \sigma_j^z \prod_{j'>j} \sigma_{j'}^x \\ \tilde{\chi}_j &\equiv \sigma_j^y \tau_{j+1/2}^z = -i\sigma_j^z \prod_{j'\geq j} \sigma_{j'}^x \end{aligned} \tag{3.2}$$

both of which are self-conjugate: $\chi_j^\dagger = \chi_j, \;\; \tilde{\chi}_j^\dagger = \tilde{\chi}_j$, so they are **majorana fermion operators**. Furthermore, they satisfies fermion commutation relations if $i \neq j$. To see the anti-commutation relation, WLOG, suppose $i < j$, we have

$$\{\chi_i, \tilde{\chi}_j\} = \{\sigma_i^z \prod_{i'>i} \sigma_{i'}^x, \; \sigma_j^y \prod_{j'>j} \sigma_{j'}^x\} = \{A \otimes B, \tilde{I} \otimes \tilde{B}\} = A \otimes \{\sigma_j^x, \sigma_j^y\} \otimes \mathbb{I}^{N-j} = 0 \tag{3.3}$$

In the same way, we have the other two anti-commutations

$$\boxed{\{\chi_i, \chi_j\} = \{\tilde{\chi}_i, \tilde{\chi}_j\} = \{\chi_i, \tilde{\chi}_j\} = 0 \qquad \forall i \neq j} \tag{3.4}$$

Therefore we see $\chi, \tilde{\chi}$ **are fermions that don't have anti-particles (self-conjugate)**.

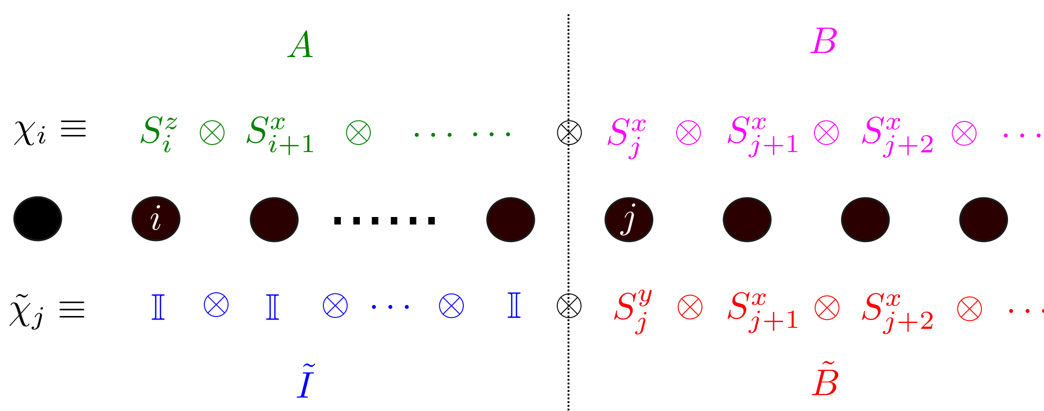

Figure 1: Visualization of the calculation of anti-commutation in Eq.(3.3)

When $i = j$ it's easy to see $\chi_i^2 = \tilde{\chi}_i^2 = 1$, so

$$\boxed{\{\chi_i, \chi_j\} = \{\tilde{\chi}_i, \tilde{\chi}_j\} = 2\delta_{ij}} \tag{3.5}$$

and that $\boxed{\{\chi_i, \tilde{\chi}_i\} = 0}$ still holds since $\{\sigma_i^x, \sigma_i^y\} = 0$. We can make sense of it by saying that they are two different flavors of fermions thus shouldn't talk to each other.

We can also make more familiar-looking objects by making complex combinations of these majoranas:

$$c_j = \frac{1}{2}(\chi_j - i\tilde{\chi}_j) \quad \Longleftrightarrow \quad c_j^\dagger = \frac{1}{2}(\chi_j + i\tilde{\chi}_j) \tag{3.6}$$

with

$$\chi_j = c_j + c_j^\dagger, \quad \tilde{\chi}_j = i(c_j - c_j^\dagger) \tag{3.7}$$

It's simple to show that they satisfies anticommutation relations:

$$\begin{aligned}\{c_i, c_j^\dagger\} &= \frac{1}{4}\{\chi_i - i\tilde{\chi}_i, \chi_j + i\tilde{\chi}_j\} = \frac{1}{4}\left(\{\chi_i, \chi_j\} + i\{\chi_i, \tilde{\chi}_j\} - i\{\tilde{\chi}_i, \chi_j\} + \{\tilde{\chi}_i, \tilde{\chi}_j\}\right) \\ &= \frac{1}{4}\left(2\delta_{ij} + i0 - i0 + 2\delta_{ij}\right) = \delta_{ij}\end{aligned} \tag{3.8}$$

$$\begin{aligned}\{c_i, c_j\} &= \frac{1}{4}\{\chi_i - i\tilde{\chi}_i, \chi_j - i\tilde{\chi}_j\} = \frac{1}{4}\left(\{\chi_i, \chi_j\} - i\{\chi_i, \tilde{\chi}_j\} - i\{\tilde{\chi}_i, \chi_j\} - \{\tilde{\chi}_i, \tilde{\chi}_j\}\right) \\ &= \frac{1}{4}\left(2\delta_{ij} - i0 - i0 - 2\delta_{ij}\right) = 0\end{aligned} \tag{3.9}$$

so for all i, j we have

$$\boxed{\{c_i, c_j^\dagger\} = \delta_{ij}, \quad \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0} \tag{3.10}$$

which defines a good fermion operator. Now, in order to write TFIM by these fermion operators, we need to figure out how to write the zz-coupling term and transverse field term. Using definitions just introduced it is simple to see that the transverse field term is

$$\sigma_j^x = -i\tilde{\chi}_j\chi_j = -i * i(c_j - c_j^\dagger)(c_j + c_j^\dagger) = c_j c_j^\dagger - c_j^\dagger c_j = -2c_j^\dagger c_j + 1 \tag{3.11}$$

note that $n_j = c_j c_j^\dagger$ can only take values 1 or 0 for occuppied or not occuppied, hence $2c_j c_j^\dagger + 1 = 1$ if $n_j = 0$; $2c_j c_j^\dagger = -1$ if $n_j = -1$. Therefore we can abbreviate the above equation as

$$\boxed{\sigma_j^x = -i\tilde{\chi}_j\chi_j = -2c_j c_j^\dagger + 1 = (-1)^{c_j^\dagger c_j}} \tag{3.12}$$

We can make sense of it by identifing left and right spin as

$$|\rightarrow_j\rangle = |n_j = 0\rangle\,, \quad |\leftarrow_j\rangle = |n_j = 1\rangle \tag{3.13}$$

i.e. the number of spin flips is the number of fermions.

The zz-coupling term is

$$\boxed{\sigma_j^z \sigma_{j+1}^z = i\tilde{\chi}_{j+1}\chi_j} \tag{3.14}$$

which can be checked by

$$i\tilde{\chi}_{j+1}\chi_j = i\left(\sigma_{j+1}^y \prod_{k \geq j+2} \sigma_k^x\right)\left(\sigma_j^z \prod_{k \geq j+1} \sigma_k^x\right) = i\sigma_{j+1}^y \sigma_j^z \sigma_{j+1}^x = \sigma_j^z \sigma_{j+1}^z.$$

So the TFIM can be written in a quadratic form:

$$\boxed{H = -J\sum_j (i\tilde{\chi}_{j+1}\chi_j - gi\tilde{\chi}_j\chi_j)} \tag{3.15}$$

This is the TFIM in majorana representation.

4 Bogoliubov transformation

First of all let us write the Hamiltonian in terms of $c_i, c_i^\dagger$ fermions. From the previous section we already know that the Hamiltonian is quadratic in majoranas, so it must also be quadratic in fermions since $\chi, \tilde{\chi}$ is a linear function of $c, c^\dagger$. The first term of majorana Hamiltonian can be written as

$$i\tilde{\chi}_{j+1}\chi_j = -(c_{j+1} - c_{j+1}^\dagger)(c_j + c_j^\dagger) = c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j^\dagger + c_j c_{j+1} \tag{4.1}$$

the second term in majorana Hamiltonian becomes

$$gi\tilde{\chi}_j\chi_j = -g(c_j - c_j^\dagger)(c_j + c_j^\dagger) = g(c_j^\dagger c_j - c_j c_j^\dagger) = 2gc_j^\dagger c_j - g \tag{4.2}$$

hence

$$H = -J\sum_j [c^\dagger_{j+1}c_j + c^\dagger_j c_{j+1} + c^\dagger_{j+1}c^\dagger_j + c_j c_{j+1} - 2gc^\dagger_j c_j + g] \tag{4.3}$$

we now take the Fourer transform

$$c_k = \frac{1}{\sqrt{N}}\sum_j c_j e^{-ikr_j}, \;\; c^\dagger_k = \frac{1}{\sqrt{N}}\sum_j c^\dagger_j e^{ikr_j} \tag{4.4}$$

$$c_j = \frac{1}{\sqrt{N}}\sum_k c_k e^{ikr_j}, \;\; c^\dagger_j = \frac{1}{\sqrt{N}}\sum_k c^\dagger_k e^{-ikr_j} \tag{4.5}$$

applying the F.T. to all terms of Hamiltonian:

$$\sum_j c^\dagger_{j+1}c_j = \frac{1}{N}\sum_{jkk'} c^\dagger_k c_{k'} e^{-ik(r_j+a)}e^{ik'r_j} = \sum_{kk'} c^\dagger_k c_{k'} e^{-ika}\frac{1}{N}\underbrace{\sum_j e^{-i(k-k')r_j}}_{N\delta_{k.k'}} = \sum_k c^\dagger_k c_k e^{-ika} \tag{4.6}$$

$$\sum_j c^\dagger_{j+1}c^\dagger_j = \frac{1}{N}\sum_{jkk'} c^\dagger_k c^\dagger_{k'} e^{-ik(r_j+a)}e^{-ik'r_j} = \sum_{kk'} c^\dagger_k c^\dagger_{k'} e^{-ika}\frac{1}{N}\underbrace{\sum_j e^{-i(k+k')r_j}}_{N\delta_{k,-k'}} = \sum_k c^\dagger_{-k} c^\dagger_k e^{ika} \tag{4.7}$$

thus their conjugate give

$$\sum_j c^\dagger_j c_{j+1} = \left(\sum_j c^\dagger_{j+1}c_j\right)^\dagger = \sum_k c^\dagger_k c_k e^{ika} \tag{4.8}$$

$$\sum_j c_j c_{j+1} = \left(\sum_j c^\dagger_{j+1}c^\dagger_j\right)^\dagger = \sum_k c_k c_{-k} e^{-ika} = \sum_k -c_{-k}c_k e^{-ika} \tag{4.9}$$

where in the last step we used $\{c_k, c_{k'}\} = 0$. Therefore the fermionic Hamiltonian becomes

$$H = J\sum_k \left[2(g-\cos ka)c^\dagger_k c_k - i\sin ka\left(c^\dagger_{-k}c^\dagger_k + c_{-k}c_k\right) - g\right] \tag{4.10}$$

which, like the majorana Hamiltonian, is also quadratic as expected. Now we can move on to apply Bogoliubov transformation and diagonalize it.

5 Continuum Limit

5.1 scale invariance

The end result of BdG diagonalization is

$$\epsilon_k = 2J\sqrt{1+g^2-2g\cos ka} \tag{5.1}$$

The energy is minimized at $k = 0$, that is

$$\epsilon_k \geq \epsilon_0 = 2J|1-g| = \Delta(g) \tag{5.2}$$

and the gap vanishes at $g = 1$ the critical point. For small k at critical point we have

$$\epsilon_k = 2J\sqrt{2(1-\cos ka)} \approx 2J\sqrt{2\times\frac{1}{2}(ka)^2} = c|k| \tag{5.3}$$

which is relativistic with speed of light $c \equiv 2Ja$. Because we are interested in the melieu of the critical field, we consider a small deviation of g_c such that $g \to g_c = 1$. Using $1 = g_c$, we have

$$\begin{aligned}\epsilon_k &\approx 2J\sqrt{1+g^2-2g(1-\frac{1}{2}k^2a^2)} = 2J\sqrt{gk^2a^2+(g-g_c)^2} \\ &= c\sqrt{gk^2+\left(\frac{g-g_c}{a}\right)^2} = c\sqrt{k^2+(g-g_c)k^2+\left(\frac{g-g_c}{a}\right)^2} \\ &\approx c\sqrt{k^2+\left(\frac{g-g_c}{a}\right)^2}\end{aligned} \tag{5.4}$$

where in the last step we neglected $(g-g_c)k^2 = O(\delta^3)$. So we can identify the mass as

$$m^2 \to 0 \quad \Longleftrightarrow \quad \lim_{g\to g_c}\left(\frac{g-g_c}{a}\right)^2.$$

that is, there a diverging length scale:

$$\xi = \frac{1}{m} = \frac{a}{|g-g_c|}.$$

so we expect the correlation length $\xi \sim |g-g_c|^{-\nu}$ has the critical exponent $\nu = 1$. Notice if we rescale space and time according to

$$x \to \lambda x \quad t \to \lambda^z t \tag{5.5}$$

with z defined by $\epsilon_k \propto k^z$, which in our case is $z = 1$, hence $\xi \to \lambda\xi,\ k \to k/\lambda, c \to \lambda c$ ($[c] = [Ja] = [kg \cdot m^3/s^2]$ $\to \lambda[Ja]$). Then the disepersion rescales to

$$\epsilon_k \to \lambda c\sqrt{k^2/\lambda^2 + m^2/\lambda^2} = c\sqrt{k^2+m^2} \tag{5.6}$$

which is invariant under rescaling.

5.2 continuum fermion field [2]

We define the continuum Fermi field

$$\Psi(x_i) = \frac{1}{\sqrt{a}}c_i \tag{5.7}$$

where a is the lattice constant. Note that c_i is dimensionless, so the normalization factor $1/\sqrt{a}$ sets the unit of field operator Ψ to be inverse square root of length, so that the Kronecker delta becomes Dirac delta in the continuous limit by $1/a\delta_{x,x'} \equiv \delta(x-x')$. The anti-commutation relations reads

$$\{\Psi(x), \Psi^\dagger(x')\} = \delta(x-x') \tag{5.8}$$

The Fourier transform becomes

$$\Psi(k) = \frac{1}{\sqrt{L}}\int dx\Psi(x)e^{-ikx} \tag{5.9}$$

where $L \equiv Na$. Now plug the $c_k \to \sqrt{1/L}\int dx\Psi(x)\exp(-ikx)$ into the fermion Hamiltonian, the first term gives

$$\begin{aligned} J\sum_k(g-\cos ka)c_k^\dagger c_k &\approx J\sum_k(g-g_c)\frac{1}{L}\int_{-\infty}^{\infty}dx\Psi^\dagger(x)e^{ikx}\int_{-\infty}^{\infty}dy\Psi(y)e^{-iky} \\ &\to J(g-g_c)\int dx\Psi^\dagger(x)\int\frac{dk}{2\pi}e^{ik(x-y)}\int dy\Psi(y) \\ &= J(g-g_c)\int dx\ \Psi^\dagger(x)\Psi(x) \end{aligned} \tag{5.10}$$

where we we assumed $ka \ll 1$ and have ignored $O(k^2)$ and higher order, and in the second row we used $Na/L = 1$ which is omitted. The second terms gives

$$\begin{aligned} -iJ\sum_k \sin ka\ c_{-k}c_k &\approx -iJa\int dx\Psi(x)\frac{Na}{L}\int\frac{dk}{2\pi}ke^{ik(x-y)}\int dy\psi(y) \\ &= -iJa\int dx\Psi(x)\frac{(-i)\partial}{\partial(x-y)}\int\frac{dk}{2\pi}e^{ik(x-y)}\int dy\Psi(y) \\ &= -Ja\int dx\Psi(x)\frac{\partial}{\partial(x-y)}\delta(x-y)\int dy\Psi(y) \\ &= -Ja\int dx\Psi(x)\delta(x-y)\frac{\partial}{\partial y}\int dy\Psi(y) = -iJa\iint dxdy\Psi(x)\delta(x-y)\partial_y\Psi(y) \\ &= -\frac{c}{2}\int dx\Psi(x)\partial_x\Psi(x) \end{aligned} \tag{5.11}$$

where we haved used $\{d/dx, \delta(x)\} = 0$. Similarly we get the other term

$$-iJ\sum_k \sin ka\ c_{-k}^\dagger c_k^\dagger \to \frac{c}{2}\int dx\Psi^\dagger(x)\partial_x\Psi^\dagger(x) \tag{5.12}$$

Therefore, the continuous limit gives the Hamiltonian

$$H \to \frac{v}{2} \int dx \left(\Psi^\dagger(x) \partial_x \Psi^\dagger(x) - \Psi(x) \partial_x \Psi(x) \right) + \Delta \int dx \Psi^\dagger \Psi \tag{5.13}$$

where $\Delta = 2J(g - g_c)$, and $v = c$ is the velocity.

6 EOM

We'd like to consider eom of the Majorana Hamiltonian:

$$H = -J \sum_l (i\tilde{\chi}_{l+1}\chi_l - gi\tilde{\chi}_l\chi_l) \tag{6.1}$$

The Heisenberg emo is given by $i\partial_t \mathcal{O} = [H, \mathcal{O}]$. Now we evaluate the commutator. Canceling i:

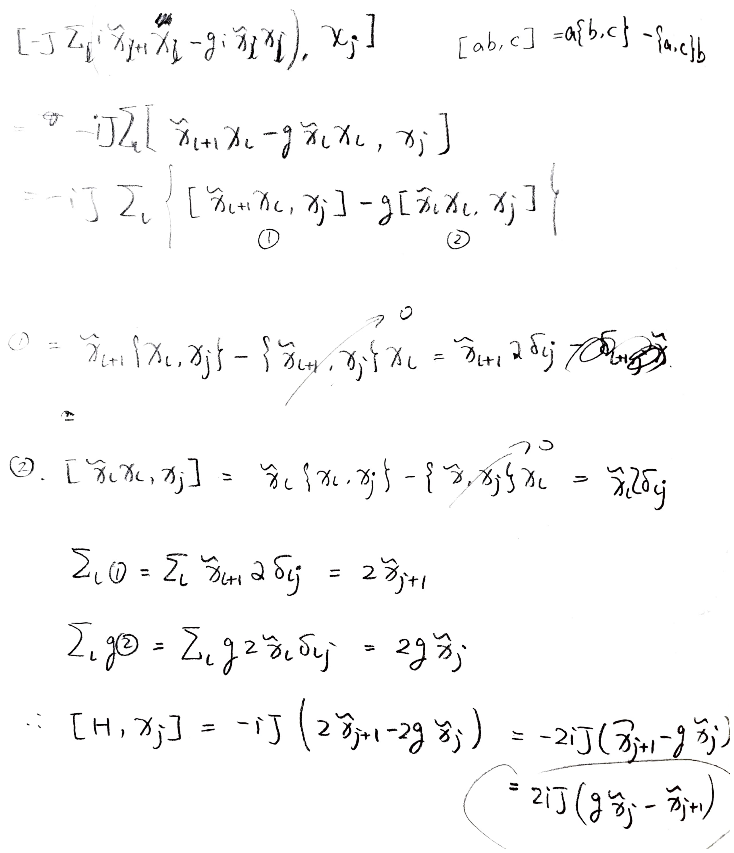

Figure 2: Derivation of Heisenberg eom

$$\begin{aligned} \partial_t \chi_j &= 2J(g\tilde{\chi}_j - \tilde{\chi}_{j+1}) \\ \partial_t \tilde{\chi}_j &= 2J(-g\chi_j + \chi_{j+1}) \end{aligned} \tag{6.2}$$

In the continuous limit we rewrite χ_{j+1} as:

$$\chi(j+1) = \chi(x_j) + a\partial_x \chi(x_j) + \mathcal{O}(a^2) \tag{6.3}$$

so we can rewrite the eom by:

$$\partial_t \chi(x) \approx 2J[g\tilde{\chi}(x) - (\tilde{\chi}(x) + a\partial_x \tilde{\chi}(x_j))].$$

that is

$$\frac{1}{2aJ} \partial_t \chi(x) \approx -\left(\frac{1-g}{a} \right) \tilde{\chi}(x) - \partial_x \tilde{\chi}(x) \tag{6.4}$$

$$\frac{1}{2aJ}\partial_t\tilde{\chi}(x) \approx +\left(\frac{1-g}{a}\right)\chi(x) - \partial_x\chi(x) \tag{6.5}$$

this can be reformed by defining $\chi_\pm = (1/2)(\tilde{\chi} \mp \chi)$, which is clear just by adding and subtracting equations. We have:

$$\frac{1}{2aJ}\partial_t\chi_- = -\partial_x\chi_- - \left(\frac{1-g}{a}\right)\chi_+ \equiv -\partial_x\chi_- - m\chi_+ \tag{6.6}$$

$$\frac{1}{2aJ}\partial_t\tilde{\chi}_+ = +\partial_x\chi_+ + \left(\frac{1-g}{a}\right) + m\chi_- = +\partial_x\chi_+ + m\chi_- \tag{6.7}$$

This gives chiral fermions at critical point $g \to 1$:

$$(\partial_0 \mp \partial_x)\chi_\pm = 0 \tag{6.8}$$

away from $g = 1$ it becomes Dirac equation with non-zero mass m.

7 Majorana Hamiltonian

In this section we rewrite the the continuous theory near at critical point in terms of majorana field. It is clear from the previous section that there are two majoranas, the left and the right mover, that propogate independently; each of them is governed by its own equation of motion. The original majorana decomposition can be rewritten by $\chi_\pm$ defined previously. For the consistency with other literature we redefine them by flipping the sign, which doesn't affect the eom:

$$\chi_+ = \frac{1}{2}(\chi - \tilde{\chi}), \;\; \chi_- = -\frac{1}{2}(\chi + \tilde{\chi}) \tag{7.1}$$

their inversion gives

$$\tilde{\chi} = -(\chi_+ + \chi_-), \;\; \chi = \chi_- - \chi_+ \tag{7.2}$$

hence

$$\Psi = \frac{1}{2}(\chi - i\tilde{\chi}) = \frac{1}{2}\left[(\chi_- + i\chi_-) - (\chi_+ - i\chi_+)\right] \tag{7.3}$$

so the first term in Eq. 5.13 gives

$$\begin{aligned}\Psi^\dagger\partial_x\Psi^\dagger = &\frac{1}{4}\left[(\chi_- - i\chi_-)\partial_x(\chi_- - i\chi_-) + (\chi_+ + i\chi_+)\partial_x(\chi_+ + i\chi_+)\right] \\ &- \frac{1}{4}\left[(\chi_- - i\chi_-)\partial_x(\chi_+ + i\chi_+) + (\chi_+ + i\chi_+)\partial_x(\chi_- - i\chi_-)\right]\end{aligned} \tag{7.4}$$

the second term in Eq. 5.13 gives

$$\begin{aligned}\Psi\partial_x\Psi = &\frac{1}{4}\left[(\chi_- + i\chi_-)\partial_x(\chi_- + i\chi_-) + (\chi_+ - i\chi_+)\partial_x(\chi_+ - i\chi_+)\right] \\ &- \frac{1}{4}\left[(\chi_- + i\chi_-)\partial_x(\chi_+ - i\chi_+) + (\chi_+ - i\chi_+)\partial_x(\chi_- + i\chi_-)\right]\end{aligned} \tag{7.5}$$

It is easy to see that the second rows of the above two equations will cancel, leaving only the first rows; furthermore, the real part ofs the terms in the first rows cancel. It is then straightforward to get

$$\Psi^\dagger\partial_x\Psi^\dagger - \Psi\partial_x\Psi = -2i\chi_-\partial_x\chi_- + 2i\chi_+\partial_x\chi_+ \tag{7.6}$$

so we can decouple the Hamiltonian into two:

$$H_- = \int dx(-iv\chi_-\partial_x\chi_-) \tag{7.7}$$

$$H_+ = \int dx(iv\chi_+\partial_x\chi_+) \tag{7.8}$$

References

[1] Continentino, M. *Quantum Scaling in Many-Body Systems: An Approach to Quantum Phase Transitions* (Cambridge University Press, 2017), 2 edn.

[2] Sachdev, S. *Quantum Phase Transitions* (Cambridge University Press, 2011), 2 edn.