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# Abelian and Non-Abelian Phases in Kitaev Honeycomb Model

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## Sketch

In last semester I learned the ground state of Kitaev's honeycomb Hamiltonian by Majorana fermionization, and a brief discussion of entanglement entropy in Toric code model, i.e. the TEE in Abelian limit, or the Toric code (TC) limit, of Kitaev honeycomb Hamiltonian.

Based on last term paper, my goal in this paper is to understand how the Abelian phase (TC) and non-Abelian phase (Gapped Phase B) in Kitaev model differ from each other. Specifically, I will (1) discuss the exact solution of Toric code model and understand its ground state. (2) review topological properties of ground state, anyon excitations, fusion rules and braiding statistics of anyons in both phases.

## 1 Ising gauge Theory

This section I will discuss the Ising gauge theory using transversal-field Ising model (TFIM) as a toy model. Of course it is a digression from the main topic, though, the purpose is to familiarize myself with the lattice gauge language and some terminology.

### 1.1 Construction of Gauge Theory from TFIM

First of all, let us construct a gauge theory on dual lattice. Consider the TFIM model in  $d = 2$  lattice:

$$H = -J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - h \sum_j \sigma_j^x \quad (1.1)$$

To build a gauge theory, some redundancy must be introduced. This can be done by defining a *dual lattice*, in contrast to the original *direct lattice*. The dual lattice has sites defined on the center of plaquettes in direct lattice, and the dofs live on links:

$$\tau_{ij}^x \equiv \sigma_i^z \sigma_j^z \quad (1.2)$$

where  $\tilde{i}\tilde{j}$  is the link on the dual lattice that intersects the link  $ij$  on the direct lattice. Note that by going to the dual lattice, the number of dofs doubles. This is because there are  $2N$  links on a lattice with  $N$  vertices (for  $N \rightarrow \infty$  or on a torus). In other words, there are  $2N$   $\tau$ s on dual lattice in contrast to  $N$   $\sigma$ s on direct lattice. However, duality is supposed to be an exact rewriting of the original theory, so we need  $N$  constraints to get rid of the redundancy.

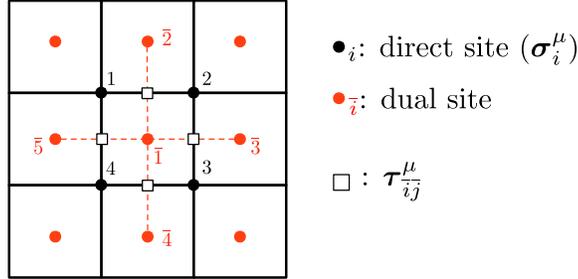


Figure 1: Dual lattice of the 2d TFIM. The dual lattice sites lie at centers of plaquettes the direct lattice, with dofs living on links.

The construction of such constraint is simple in this toy model. The overall idea is that  $\tau$ s must satisfy the same Pauli algebra defined on the direct lattice. Consider a set of 4 links that emanate from a single site on the dual lattice, i.e. four links that bound a plaquette on the direct lattice. Their product is:

$$\tau_{\tilde{1}\tilde{2}}\tau_{\tilde{1}\tilde{3}}\tau_{\tilde{1}\tilde{4}}\tau_{\tilde{1}\tilde{5}} = (\sigma_1^z\sigma_2^z)(\sigma_2^z\sigma_3^z)(\sigma_3^z\sigma_4^z)(\sigma_4^z\sigma_1^z) = 1 \quad (1.3)$$

Therefore the product of the four  $\tau^x$  is constrained to be 1. Such local constraint is defined by the loop around a vertex of the dual lattice, hence, the total number of constraints is  $N$ , which exactly matches our need to recover the physical space. To write it compactly:

$$\prod_{+} \tau_{\tilde{i}\tilde{j}}^x = 1 \quad \text{for all dual lattice sites} \quad (1.4)$$

Now we have rewritten the coupling term of TFIM in terms of  $\tau^x$ , the next step is to figure out how to rewrite the flipping term  $\sigma^x$  in the original model. In 2d, flipping a spin on  $i$  in direct lattice corresponds to exiting 4 domain walls around the site, which forms a plaquette on the dual lattice. Note that the domain wall operators  $\tau$  also live in Pauli space, to flip eigen states of  $\tau^x$  on a plaquette, we simply apply 4  $\tau^z$ s:

$$\sigma_i^x \Leftrightarrow \prod_{\tilde{i}\tilde{j} \in \diamond} \tau_{\tilde{i}\tilde{j}}^z \quad (1.5)$$

Therefore the gauge theory of TFIM is written as:

$$H = -J \left( \sum_{\langle ij \rangle} \tau_{ij}^x + g \sum_{\diamond} \prod_{ij \in \diamond} \tau_{ij}^z \right) \quad (1.6)$$

with  $h = g/J$  and constraint by:

$$\prod_{+} \tau_{ij}^x = 1 \quad (1.7)$$

Now let us consider the gauge theory introduced above as a fundamental model (to emphasize this, we'll stop using 'barred' indices, to remind ourselves that the  $\tau$ s are now the 'direct' variables. For a pure gauge theory, the fundamental Hamiltonian is:

$$H = -J \left( \sum_{\langle ij \rangle} \tau_{ij}^x + g \sum_{\diamond} \prod_{ij \in \diamond} \tau_{ij}^z \right) \quad (1.8)$$

with the gauge transformation defined as:

$$G_i = \prod_{+} \tau_{ij}^x \quad (1.9)$$

in which all links  $ij$  emanate from site  $i$ .  $G_i$  can be viewed as a local  $Z_2$  gauge transformation operator. This can be shown by  $G_i^\dagger \tau_l^z G_i = \zeta \tau_l^z$ , where  $\zeta = -1$  only if the site  $i$  is at the end of link  $l$ , and  $\zeta = 1$  otherwise.

It is readily to check the gauge transformation operator commute with Hamiltonian ( $G_i$  and  $H$  share at most 2 common links that produce terms with oppsite signs):

$$[H, G_i] = 0 \quad (1.10)$$

From the definition in Eq.(1.9)  $G_i$  clearly obeys  $G_i^2 = 1$ , so that an energy eigen state of  $H$   $|\psi\rangle$  must satisfy  $G_i |\psi\rangle = \pm |\psi\rangle$ . If we are to view the original lattice as the fundamental, then our pure gauge theory must have a conserved  $G_i$ , i.e.  $G_i = 1$ . In other words, any quantum state  $|\psi\rangle$  generated by the gauge Hamiltonian is physical only if it satisfies:

$$G_i |\psi\rangle = |\psi\rangle \quad (1.11)$$

We say that the state has a gauge symmetry where the gauge transformation evaluates to 1 on every state in the physical Hilbert space.

## 1.2 Confined Phase

Now we turn to the phase diagram of Ising gauge theory. First of all we look at the weak coupling limit i.e.  $g \ll 1$ . The Hamiltonian becomes:

$$H \simeq -J \sum_{\langle ij \rangle} \tau_{ij}^x \quad \text{with} \quad G_i = 1 \quad (1.12)$$

The ground state is simply  $|g.s.\rangle = |\{\tau_{ij}^x = 1\}\rangle$  which satisfies the constraint  $G_i = 1$  as well.

On adding a small but non-zero  $g$ , some of the links will be excited to  $\tau_{ij}^x = -1$ , thus equivalent to some links that have nonzero electric fields; of course, to satisfy the constraint, the field lines must form closed loops (in the spin language, there are a few domain walls) This is similar to the fact that Ising-ordered phase does not have all spins up or down — we can have a few domains of minority spins without becoming paramagnetic. In other words, the electric field lines are confined.

### 1.3 Deconfined Phase

This is the opposite limit compared to confined phase, where  $g \gg 1$ :

$$H_{g=\infty} = -Jg \sum_{\diamond} \prod_{\diamond} \tau_{ij}^z, \quad \text{with } G_i = 1 \quad (1.13)$$

Note that the magnetic flux operator  $\prod_{\diamond} \tau_{ij}^z$  commutes with Hamiltonian, so the ground state occurs when magnetic flux through each plaquette is trivial:

$$\prod_{\diamond} \tau_{ij}^z = +1 \quad \text{for all } \diamond \quad (1.14)$$

Now we construct the ground state. Let's start with the trivial configuration in  $\tau_z$  basis, where  $|\Psi_0\rangle = \bigotimes_{ij} |\tau_{ij}^z = 1\rangle$ . Note that this wavefunction is not gauge invariant since apparently  $G_i$  will flip spins on 4 links thus  $G_i |\Psi_0\rangle \neq |\Psi_0\rangle$ . Such a local gauge transformation can be fixed by redefining our wavefunction:

$$|\Psi\rangle = |\Psi_0\rangle + G_i |\Psi_0\rangle \quad (1.15)$$

such that

$$G_i |\Psi\rangle = G_i |\Psi_0\rangle + G_i^2 |\Psi_0\rangle = G_i |\Psi_0\rangle + |\Psi_0\rangle \quad (1.16)$$

which is invariant under transformation by  $G_i$ . By taking all local gauge transformations into consideration, the exact ground state is:

$$|g.s.\rangle = \prod_i (1 + G_i) |\Psi_0\rangle \quad (1.17)$$

since  $[H, G_i] = 0$ , the wavefunction above has the same energy as  $|\Psi_0\rangle$ . Essentially, we're superposing all gauge-equivalent wavefunction into one gauge-equivalent class. In a gauge theory, different physical states are characterised by different gauge-equivalent classes.

## 2 Toric Code Model

Toric code Hamiltonian has dofs living on links, such that:

$$H_{TC} = -J_m \sum_p B_p - J_e \sum_s A_s \quad (2.1)$$

where  $B_p$  and  $A_s$  are the plaquette operator and star operators shown in Fig.(2):

$$B_p = \prod_{\diamond} \tau_l^z, \quad A_s = \prod_{+} \tau_l^x \quad (2.2)$$

Note that there's no constraint on this Hamiltonian, and is written purely in

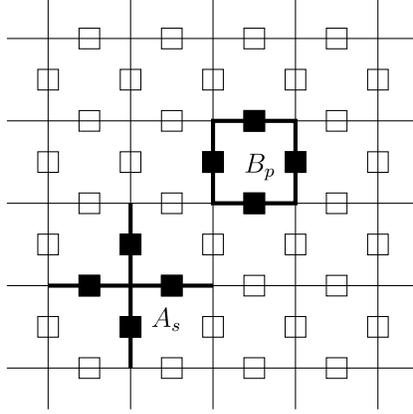


Figure 2:  $B_p$  and  $A_s$  are the plaquette operator and star operators defined on square lattice

terms of local dofs.

## 2.1 Ground State Wavefunction

The key of the construction of ground state is noting that the Hamiltonian is made up of purely commuting terms. It's simple to check that:

$$\begin{aligned} [A_s, A_{s'}] &= 0 \\ [B_p, B_{p'}] &= 0 \\ [A_s, B_p] &= 0 \end{aligned} \quad (2.3)$$

so that both plaquette and star operators commute with Hamiltonian:

$$[A_s, H] = [B_p, H] = 0 \quad (2.4)$$

so that  $A_s$  and  $B_p$  can be simultaneously diagonalized. Assuming  $J > 0$ , the ground state is when all  $B_p = 1$  and  $A_s = 1$ . However, we would like to write the ground state in the fundamental dofs, i.e. the links. This can be done by introducing the classical variable  $s_l = \pm 1$  in  $\tau_z$  basis. The ground state is some superposition of vortex-free configurations. We must have:

$$B_p |\psi_0\rangle = |\psi_0\rangle \quad \Rightarrow \quad |\psi_0\rangle = \sum_{v.f.} c_s |s\rangle \quad (2.5)$$

Note that  $A_s$  is a good quantum number, which evaluates to +1 at g.s.

$$A_s |\psi_0\rangle = |\psi_0\rangle \quad (2.6)$$

This condition holds true if and only if all the  $c_s$  are equal for each orbit of the  $A_s$ . Hence the ground state wavefunction is an equal-weight superposition of vortex-free configuration.

We can also construct the ground state from the gauge point of view. We can view  $A_s$  as a gauge transformation operator. A physical states must satisfy:

$$A_s |\Psi_0\rangle = |\Psi_0\rangle \quad (2.7)$$

We can choose to start with the trivial  $|\Psi_0\rangle = \bigotimes_l |s_l = 1\rangle$ , which is not gauge invariant since apparently  $A_s$  will flip spins on 4 links thus  $A_s |\Psi_0\rangle \neq |\Psi_0\rangle$ . Such a local gauge transformation can be fixed by redefining our wavefunction:

$$|\Psi\rangle = |\Psi_0\rangle + A_s |\Psi_0\rangle \quad (2.8)$$

such that

$$A_s |\Psi\rangle = A_s |\Psi_0\rangle + A_s^2 |\Psi_0\rangle = A_s |\Psi_0\rangle + |\Psi_0\rangle \quad (2.9)$$

now we have acquire the gauge invariant wavefunction. In this way, we write the ground state as:

$$|\Psi\rangle \propto \prod_s (1 + A_s) |\Psi_0\rangle \quad (2.10)$$

Essentially, we're superposing all gauge-equivalent wavefunction into one gauge-equivalent class.

## 2.2 Ground State Degeneracy

The Degeneracy of TC ground state is topological, in which any local operator has vanishing off-diagonal matrix elements between them in the thermo-dynamic limit. Therefore there is each dimension in the ground state is locally a "superselection" regime. We can, however, define non-local operators on a torus i.e. Wilson loops, as shown in Fig.(3)

Such Wilson loop operators transform the ground state from one of the degenerate state to the other. On the torus, we define Wilson loops as:

$$W_{x/y}^e = \prod_{j \in C_{x/y}^e} \tau_j^z \quad (2.11)$$

$$W_{x/y}^m = \prod_{j \in C_{x/y}^m} \tau_j^x \quad (2.12)$$

where  $C_{x/y}^e$  is a set of spins on bond parallel to the non-contractable loops in  $x$  or  $y$  direction, and  $C_{x/y}^m$  is a set of spins on bonds perpendicular to the non-contractable loop that connects the centers of plaquettes in  $x$  or  $y$  direction. It's simple to check the following commutation relations:

$$\begin{aligned} [W_{x/y}^e, A_s] &= [W_{x/y}^e, B_p] = 0 \\ [W_{x/y}^m, A_s] &= [W_{x/y}^m, B_p] = 0 \end{aligned} \quad (2.13)$$

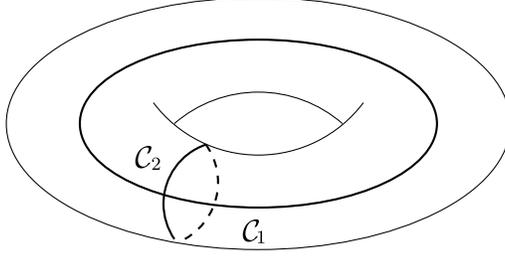


Figure 3: The path of non-local Wilson loop operators on a torus.

so that

$$[W_{x/y}^{e/m}, H] = 0 \quad (2.14)$$

Furthermore, due to the anti-commutation relation of  $\tau_x$  and  $\tau_z$ , different types of Wilson loops anti-commute if they are perpendicular to each other:

$$\{W_x^e, W_y^m\} = 0 \quad (2.15)$$

Now we are going to show that such anti-commutation relation necessarily bring degeneracy in the energy spectrum. Let  $|\Psi\rangle$  be an eigen state of Hamiltonian with energy  $E_\Psi$ . Due to Eq.(2.14), we have:

$$HW_x^e |\Psi\rangle = E_\Psi W_x^e |\Psi\rangle \quad (2.16)$$

so that  $W_x^e |\Psi\rangle$  is also an eigen state. Same is true for  $W_y^m$ . Now there are two possible cases, that whether or not  $W_{x/y}^{e/m}$  leave the eigen state  $|\Psi\rangle$  unchanged. Suppose wavefunction is not effected by the Wilson loops, i.e.

$$W_x^e |\Psi\rangle = c |\Psi\rangle = |\Psi\rangle \quad (2.17)$$

where  $c$  must be 1 since  $(W_{x/y}^{e/m})^2 = 1$ . so that:

$$W_y^m W_x^e |\Psi\rangle = W_y^m |\Psi\rangle \quad (2.18)$$

from the anti-commutator we have:

$$W_x^e W_y^m |\Psi\rangle = -W_y^m W_x^e |\Psi\rangle = -W_y^m |\Psi\rangle \quad (2.19)$$

notice that  $W_y^m |\Psi\rangle$  is also an eigen state. Such an eigen state, according to our assumption Eq.(2.17), must remain unchanged:

$$W_x^e W_y^m |\Psi\rangle = W_y^m |\Psi\rangle \quad (2.20)$$

compare Eq.(2.19) and Eq.(2.20) we have  $1 = -1$ , which is impossible. Therefore our assumption in Eq.(1.28) is wrong, the oppsite is true. This applies to both types of Wilson loop:

$$W_{x/y}^{e/m} |\Psi\rangle = |\Psi'\rangle \neq |\Psi\rangle \quad (2.21)$$

The algebra must hold in any eigen subspace of the Hamiltonian. Therefore We conclude that all eigen subspace of the Hamiltonian, including the ground state space, must be degenerate. In Other words, the off-diagonal elements of Wilson loops must NOT be zero, that one of the degenerate states can tunnel to the other only through global operation.

### 2.3 Excitations

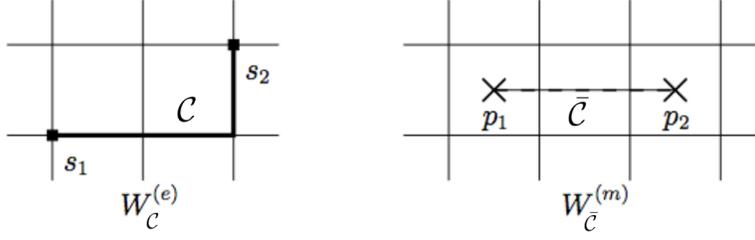


Figure 4: Electric and magnetic path operators. They create a pair of excitations at the ends of the string

#### 2.3.1 Charge Excitation

The low-lying excitations of the toric code come in two varieties, that can be identified with the electric charges and magnetic vortices of a  $Z_2$  gauge theory. We define an electric path operator, as is shown in Fig.(4):

$$W_C^{(e)}(s_1, s_2) = \prod_{l \in C} \tau_l^z \quad (2.22)$$

the path connects 2 vertices  $s_1$  and  $s_2$ . It's apparent that:

$$[W_C^{(e)}, B_p] = 0 \quad (2.23)$$

Note that  $W_C^{(e)}$  commutes with most but not all star operators. The exception occurs when the star operator is at the end points of electric path  $C$ , which we label  $A_{s_1}$  and  $A_{s_2}$ . On the shared link, The  $\sigma_z$  operator in electric path anti-commutes with the  $\sigma_x$  of  $A_s$ . so that:

$$\begin{aligned} W_C^{(e)}(s_1, s_2) A_{s_1} &= -A_{s_1} W_C^{(e)}(s_1, s_2) \\ W_C^{(e)}(s_1, s_2) A_{s_2} &= -A_{s_2} W_C^{(e)}(s_1, s_2) \end{aligned} \quad (2.24)$$

Let Eq.(2.24) act on the ground state wavefunction, by gauge invariance:

$$W_C^{(e)}(s_1, s_2) |\Psi_0\rangle = -A_{s_1} W_C^{(e)}(s_1, s_2) |\Psi_0\rangle \quad (2.25)$$

same is true for  $A_{s_2}$ . It's convenient to define:  $W_C^{(e)}(s_1, s_2) |\Psi_0\rangle \equiv |\Psi_{s_1, s_2}\rangle$ , such that:

$$\begin{aligned} |\Psi_{s_1, s_2}\rangle &= -A_{s_1} |\Psi_{s_1, s_2}\rangle \\ |\Psi_{s_1, s_2}\rangle &= -A_{s_2} |\Psi_{s_1, s_2}\rangle \end{aligned} \quad (2.26)$$

Therefore  $|\Psi_{s_1, s_2}\rangle$  is an energy eigen state of  $H_{TC}$  with energy  $4J_e$  above the ground state. We interpret this as a state with ‘charges’ (e-particles) at the sites  $s_1$  and  $s_2$ , each cost  $2J_e$  to create.

### 2.3.2 Magnetic Vortices

Define an ”magnetic path”  $W_C^{(m)}(p_1, p_2)$ :

$$W_C^{(m)}(p_1, p_2) = \prod_{l \in C} \tau_l^x \quad (2.27)$$

where  $p_1$  and  $p_2$  are labels of plaquettes, and path  $C$  is path on dual lattice (centers of the grid). A link belongs to path  $C$  is and only if it is intersected by it. In the same way, we have the commutation relation:

$$[W_C^{(m)}, A_s] = 0 \quad (2.28)$$

and All but two plaquette operators  $B_{p_1}$  and  $B_{p_2}$  at the ends of path  $C$  commute with  $W_C^{(m)}$ . Similarly, because of the anti-commutation of  $\sigma_z$  and  $\sigma_x$  at the ends:

$$B_{p_1} |\Psi_{p_1, p_2}\rangle = -|\Psi_{p_1, p_2}\rangle \quad (2.29)$$

so that the plaquette  $B_{p_1}$  in Hamiltonian raises the ground state energy by  $4J_m$ . We interpret this as a state with ‘magnetic fluxes’ (m-particles) at the plaquettes  $p_1$  and  $p_2$ , each costs  $2J_m$  to create.

## 2.4 Anyon Statistics in Toric Code

### 2.4.1 Mutual Statistics and Fusion Rules

Let consider taking a charge excitation  $e$  around a magnetic vortex excitation  $m$ . Let  $|\xi\rangle$  be a state containing a magnetic vortex at  $p_1$ . Let  $C$  be a closed loop around  $p_1$ , then the braiding operation is defined as:

$$\left( \prod_{l \in C} \tau_l^z \right) |\xi\rangle = \left( \prod_{p \in \partial C} B_p \right) |\xi\rangle \quad (2.30)$$

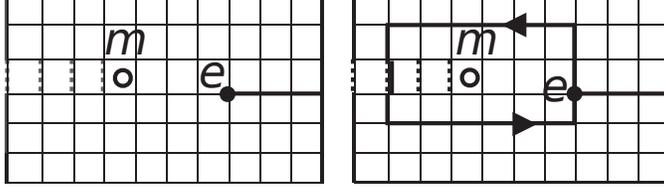


Figure 5: Start with a pair of  $m$  particles and  $e$  particles created by their own excitation operators. To make the operation well-defined we assume particles of each pair are separated with a large enough distance. Taking one of the  $e$  particle around a loop that encloses a  $m$  particle generates a phase in wavefunction

where on the R.H.S. is a lattice version of Stokes' theorem. Pictorially it this operation is shown below: Note that we have shown in the previous section that:

$$B_{p_1} |\xi\rangle = -|\xi\rangle \quad (2.31)$$

so that

$$\left( \prod_{p \in \partial C} B_p \right) |\xi\rangle = -|\xi\rangle \quad (2.32)$$

hence we see that upon braiding  $e$  around  $m$ , wavefunction changes by  $|\xi\rangle \rightarrow -|\xi\rangle$ , i.e. we pick up a phase of  $\pi$ . Since  $e$  and  $m$  are bosons, the composite particle  $f = e \times m$  is a fermion. This is the first fusion rule:

$$e \times m = f \quad (2.33)$$

Consider when two  $e$ -type excitations are moved to the same star, the loop  $C_e$  that connects them becomes a closed loop, thus the state returns to ground state. This is because there are no "dangling bonds" in the path that a plaquette operator share either 0 or 2 links with  $C_e$ , hence we recover the commutation relation. Therefore we have the fusion rule:

$$e \times e = 1 \quad (2.34)$$

where 1 stands for the ground state or vacuum. In the same way, moving two  $m$ -type excitations to the same plaquette creates a closed loop  $C_m$ , with which all stars and plaquettes commute. so the second fusion rule is:

$$m \times m = 1 \quad (2.35)$$

Note that in all of the arguments above we have moved the particles on contractible loops only. If we were to create a pair of  $e$  or  $m$  type particles and take one of the pair excitation to go around a non-contractable loop, we will be

effectively apply an Wilson loop operator  $W_{x/y}^{e/m}$  on the manifold of topologically degenerate ground states.

From these we can now conclude the braiding statistics using a pictorial scheme, with the worldline moving upwards. The braiding relation of two  $e$  particles, as well as  $m$  particle, obeys usual boson statistics, while the braiding relation between  $e$  and  $m$  does not.

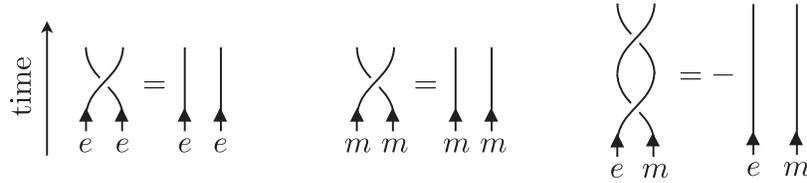


Figure 6: The braiding relation of two  $e$  particles, two  $m$  particles, and between  $e$ - $m$  particles

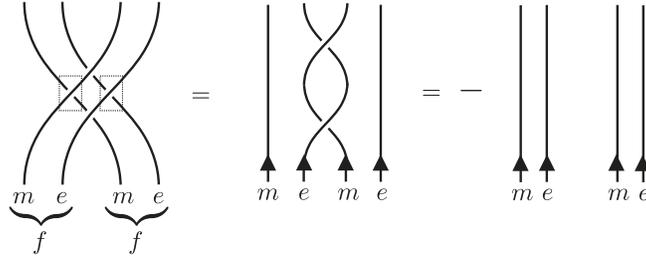


Figure 7: The braiding relation of two  $f = e \times m$  composite particles. One exchange brings a phase factor of  $-1$ . This is done by splitting the braiding of  $f$  particle into the braiding relations defined in the previous figure.

#### 2.4.2 Degeneracy from Mutual Statistics

We can also explain the ground state degeneracy on the torus by mutual statistics of excitations. First of all, let us discuss the connection between Wilson loops and mutual statistics.

Suppose there are at least one particle types e.g.  $e$  or  $m$ , with mutual statistics  $\neq 1$ . Let us define an abstract operator  $\mathcal{Z}$  that creates a pair of an excitation, takes one of the particle around a non-contractable loop  $\mathcal{C}_1$  and annihilates the pair. Similarly we can define another operator  $\mathcal{X}$  that creates another pair of excitations, takes one of them around the other non-contractable loop  $\mathcal{C}_2$  and annihilates the pair. Each of these operators preserves the ground state of the

system, i.e. the state with no excited quasi-particles ( $[\mathcal{X}, H] = [\mathcal{Z}, H] = 0$ ). If these two operators anti-commute, we have a fermionic mutual statistics; if commute, a bosonic mutual statistics:

$$\mathcal{Z}^{-1}\mathcal{X}^{-1}\mathcal{Z}\mathcal{X} = \pm 1 \quad \text{for boson and fermion} \quad (2.36)$$

In general,  $\mathcal{Z}$ ,  $\mathcal{X}$  don't necessarily (anti)commute, the result could be an arbitrary phase such that:

$$\mathcal{Z}^{-1}\mathcal{X}^{-1}\mathcal{Z}\mathcal{X} = e^{-i2\theta} \quad (2.37)$$

which stands for a generic abelian statistics.

We can explain such statement by the following. First, the process defined by  $\mathcal{X}^{-1}\mathcal{X}$  is topologically equivalent a straight line in spacetime, in conformity with that fact  $\mathcal{X}^{-1}\mathcal{X} = 1$ .

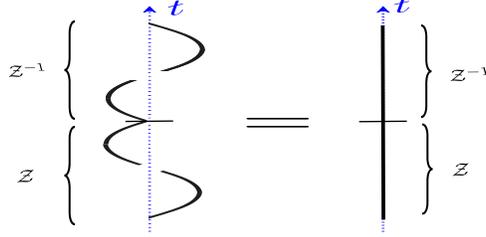


Figure 8:  $\mathcal{Z}^{-1}\mathcal{Z}$  is topologically equivalent a trivial straight line in spacetime

The spacetime topology can be shown as that one anyon world line sweeps around the loop  $\mathcal{C}_1$  and then immediately traverses the same path in the reverse order, thus can be smoothly deformed to a topologically trivial path. Same is also true for  $\mathcal{Z}^{-1}\mathcal{Z}$ .

In a similarly way, the operator  $\mathcal{Z}^{-1}\mathcal{X}^{-1}\mathcal{Z}\mathcal{X}$  can be deformed smoothly to one in which two pairs of anyons are created, one of the anyons winds clockwise to the other and both pairs annihilate themselves. The winding-around is equivalent to two successive exchanges. If we assume the exchange of two particles brings a phase  $e^{-i\theta}$ , then two exchanges will be accompanied by a total phase change  $e^{-2i\theta}$ , in keeping with Eq.(2.37). Particularly, if  $[\mathcal{X}, \mathcal{Z}] = 0$  (or  $\{\mathcal{X}, \mathcal{Z}\} = 0$ ), i.e.  $\theta = 0(\pi)$ , then we have bosonic(fermionic) statistics.

With the argument above, we're ready to look into the degeneracy. Since  $\mathcal{Z}$  and  $\mathcal{X}$  both commute with Hamiltonian, applying these operators to an eigenvector must result in the same eigenspace labeled by the same eigenvalue, i.e. they preserve the eigenspace. However, assuming  $\mathcal{Z}$  and  $\mathcal{X}$  do not commute, they cannot be diagonalized simultaneously. Assume  $|\alpha\rangle$  is an eigenvector of  $H$ :

$$\mathcal{X}|\alpha\rangle = e^{i\alpha}|\alpha\rangle \quad (2.38)$$

according to Eq.(2.37):

$$\mathcal{X}\mathcal{Z}|\alpha\rangle = e^{i2\theta}\mathcal{Z}\mathcal{X}|\alpha\rangle = e^{2i\theta}e^{i\alpha}\mathcal{Z}|\alpha\rangle \quad (2.39)$$

Suppose that  $\theta \in (0, 2\pi)$  and is a rational multiple of  $2\pi$ , such that:

$$\theta = \pi p/q \quad \text{with } p < 2q \quad (2.40)$$

where  $q, p$  are integers with no common factor other than  $p, q$  themselves. Then the eigenvalue corresponding to the eigenvector  $\mathcal{Z}|\alpha\rangle$  is:

$$\exp\left[\alpha + \left(\frac{2\pi p}{q}\right)k\right] \pmod{2\pi} \quad (2.41)$$

we then conclude that  $\mathcal{X}$  must have at least  $q$  distinct eigenvalues, i.e.  $q$  super-selective degenerate sectors. In the Toric code Hamiltonian, the  $e$ -type and  $m$ -type particle anticommute, so that the exchange factor is  $\theta = \pi/2$ , so that both  $W^e$  and  $W^m$  have 2 distinct eigenvalues, thus 4-fold degenerate in total.

### 3 Honeycomb Model

#### 3.1 Introducing a Gap by Breaking TR

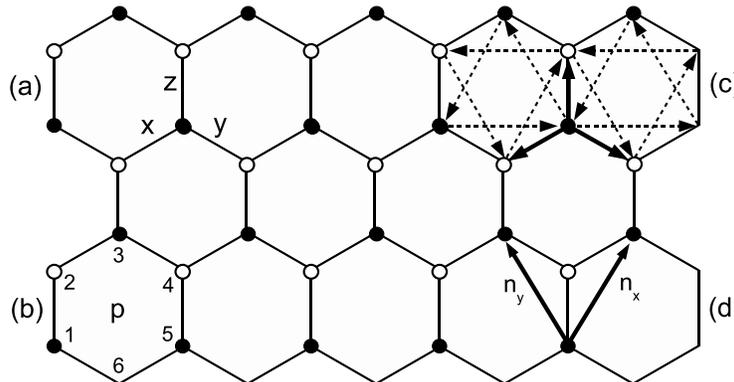


Figure 9: The honeycomb lattice. (a) anisotropic coupling in  $x, y, z$  bonds. (b) labeling of sites in a plaquette  $w_p$ . (c) The TR-breaking couplings are next-nearest neighbor couplings indicated by dashed lines. (d) unit vectors of B.L.

In Phase B there are two types of elementary excitations: gapped vortices and gapless fermions. Vortices in this phase do not have well-defined statistics, i.e. the effect of winding one vortex around the other depends on details of the process. However, we can open a gap in the gapless phase by applying a perturbation that breaks the time-reversal(TR) symmetry. Thereofre we reformulate

the Hamiltonian as:

$$\begin{aligned}
H = & -J_x \sum_{x\text{-links}} \sigma_i^x \sigma_j^x - J_y \sum_{y\text{-links}} \sigma_i^y \sigma_j^y - J_z \sum_{z\text{-links}} \sigma_i^z \sigma_j^z \\
& + K \sum_{(i,j,k) \in p} \sigma_i^x \sigma_j^y \sigma_k^z
\end{aligned} \tag{3.1}$$

where the last term being the perturbation that breaks the TR:

$$\begin{aligned}
\sum_{(i,j,k) \in p} \sigma_i^x \sigma_j^y \sigma_k^z = & \sigma_1^x \sigma_2^y \sigma_3^z + \sigma_2^x \sigma_3^y \sigma_4^z + \sigma_3^x \sigma_4^y \sigma_5^z \\
& + \sigma_4^x \sigma_5^y \sigma_6^z + \sigma_5^x \sigma_6^y \sigma_1^z + \sigma_6^x \sigma_1^y \sigma_2^z
\end{aligned} \tag{3.2}$$

Next fermionize spins into majoranas by:

$$\sigma_i^\alpha = i b_i^\alpha c_i, \quad D_i = b_i^x b_i^y b_i^z c_i \equiv 1 \tag{3.3}$$

furthermore define link operators which has eigenvalue  $u_{ij} = \pm 1$ :

$$\hat{u}_{ij} = i b_i^\alpha b_j^\alpha \tag{3.4}$$

in which  $\alpha = x, y, z$  that depends on the type of link  $(ij)$ . Therefore we can rewrite the coupled spins as:

$$\sigma_i^\alpha \sigma_j^\alpha = -i \hat{u}_{ij} c_i c_j \quad \text{and} \quad \sigma_i^x \sigma_j^y \sigma_k^z = -i \hat{u}_{ik} \hat{u}_{jk} D_k c_i c_j \tag{3.5}$$

so that we can write the Hamiltonian compactly as:

$$H = \frac{i}{4} \sum_{ij} \hat{A}_{ij} c_i c_j \tag{3.6}$$

with redefined  $\hat{A}_{ij}$ :

$$\hat{A}_{ij} = 2J_{ij} \hat{u}_{ij} + 2K \sum_k \hat{u}_{ik} \hat{u}_{jk} \tag{3.7}$$

where the first term in  $\hat{A}_{ij}$  stands for n.n. coupling between  $c_i$  and  $c_j$ , whiel the second term in  $\hat{A}_{ij}$  stands for next-to-nearest neighbor interactions. Upon fixing  $u_{ij}$ , all  $b$ -operators are removed completely, thus  $A_{ij}$  in the Hamiltonian are just a set of real numbers:

$$H = H_1 + H_2 \equiv \sum_{ij} 2J u_{ij} c_i c_j + \frac{i}{4} \sum_{ij} 2K \sum_k u_{ik} u_{jk} c_i c_j \tag{3.8}$$

the Hamiltonian then is reduced to a hopping model of majoranas. Let us now work in the vortex-free sector and set bonds to be isotropic so that  $J_{x,y,z} = J$ . First, transform  $H_1$  to momentum space by:

$$\begin{aligned}
c_i = \frac{1}{\sqrt{N}} \sum_{\vec{k}} e^{i\vec{k} \cdot \vec{r}_i} a_{\vec{k}}; \quad a_{\vec{k}} = \frac{1}{\sqrt{N}} \sum_{i \in A} e^{-i\vec{k} \cdot \vec{r}_i} c_j \\
c_j = \frac{1}{\sqrt{N}} \sum_{\vec{k}} e^{i\vec{k} \cdot \vec{r}_j} b_{\vec{k}}; \quad b_{\vec{k}} = \frac{1}{\sqrt{N}} \sum_{j \in B} e^{-i\vec{k} \cdot \vec{r}_j} c_k
\end{aligned} \tag{3.9}$$

which has been done in last year's term-paper:

$$\tilde{H}_1 = \frac{1}{4} \sum_{\vec{k}} f(\mathbf{k}) \tilde{a}_{\vec{k}}^\dagger \tilde{b}_{\vec{k}} + h.c. \quad (3.10)$$

where we have defined  $\tilde{a} = e^{-i\pi/4}$ ,  $\tilde{b} = e^{i\pi/4}$ , and  $f(\mathbf{k}) = 2J \sum_{\alpha=1,2,3} e^{i\mathbf{k}\cdot\mathbf{s}_\alpha}$ . In a similarly way, transform  $H_2$  into momentum space:

$$H_2 = \frac{1}{4} \sum_{\mathbf{k}} \Delta(\mathbf{k}) (\tilde{a}_{\mathbf{k}}^\dagger \tilde{a}_{\mathbf{k}} - \tilde{b}_{\mathbf{k}}^\dagger \tilde{b}_{\mathbf{k}}) \quad (3.11)$$

where  $\Delta(\mathbf{k})$  is defined as:

$$\Delta(\mathbf{k}) = 4K (-\sin(\mathbf{k} \cdot \mathbf{n}_1) + \sin(\mathbf{k} \cdot \mathbf{n}_2) + \sin(\mathbf{k} \cdot (\mathbf{n}_1 - \mathbf{n}_2))) \quad (3.12)$$

combine  $H_1$  and  $H_2$ , the Hamiltonian becomes:

$$H = \frac{1}{4} \sum_{\mathbf{k}} (\tilde{a}_{\mathbf{k}}^\dagger \quad \tilde{b}_{\mathbf{k}}^\dagger) \begin{bmatrix} \Delta(\mathbf{k}) & f(\vec{k}) \\ f^*(\vec{k}) & -\Delta(\mathbf{k}) \end{bmatrix} (\tilde{a}_{\mathbf{k}} \quad \tilde{b}_{\mathbf{k}}) \quad (3.13)$$

The eigen energy is then:

$$E(\mathbf{k}) = \pm \sqrt{\Delta(\mathbf{k})^2 + f(\mathbf{k})^2} \quad (3.14)$$

which has a finite gap  $\Delta(\mathbf{k})$  above ground state in contrast to non-perturbative

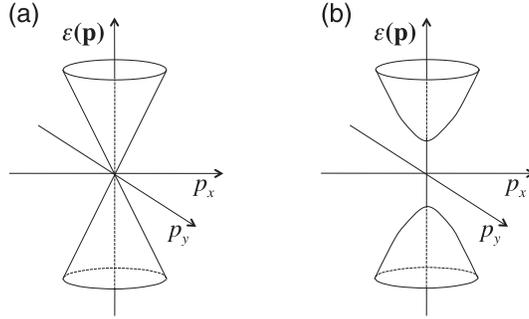


Figure 10: (a) The dispersion of non-perturbative model, fermionic excitation is gapless. (b) Upon breaking TR, a gap  $\Delta$  is opened in the gapless phase.

Hamiltonian. The ground state energy for half-filling is thus given by:

$$E_0 = - \sum_{k_x} \sum_{k_y} |E(\mathbf{k})| \quad (3.15)$$

### 3.2 Transport Vortices

In order to understand the relation between vortices and gapped fermionic excitations, in this section we study how to transport vortices and move between different vortex sectors. The vortex configurations  $\{w_p\}$  is determined by gauge field  $\{u_{ij}\}$  on links. Hence, we can change the link configuration  $\{u_{ij}\}$  in order to manipulate vortex configuration  $\{w_p\}$ . This can be done simply by changing signs of eigenvalues of link operators:  $u_{ij} \rightarrow -u_{ij}$ . However, changing gauge field directly is not physically doable. This difficulty can be addressed by noting that the  $u_{ij}$  are always paired with local couplings  $J_{ij}$  and  $K_{ijk}$ . Therefore, changing the gauge configurations  $u_{ij}$  is equivalent to tuning these couplings  $\{J, K\}$ , hence allows us to move to different vortex sectors, as well as transport vortices in a continuous way.

As an example, we can tuning  $J_z = -1$  on the  $d$  successive  $z$ -links that lie on links crossed by a string, which is shown in Fig.(11). This is equivalent to setting  $u_{ij} = -1$  on these links. This operation creates two vortices separated by distance  $d$ , since only two ends of the string create vortex excitations. Then by varying the distance  $d$  we can probe the interaction of two vortices by studying the evolution of energy spectrum as a function of separation  $d$ . Moreover, we can define a 'continuous' transport of vortex by reversing the sign of  $J_{ij}$  continuously, i.e.  $J_{ij} \equiv 1 - 2s/S$  with  $s \in [0, S]$ . Under such tuning protocol, the energy spectrum can evolve smoothly between different vortex sectors.

Using manipulation defined above, we can evaluate the fermionic energy spectrum of the Hamiltonian with zero and two vortices. See in Fig.(12).

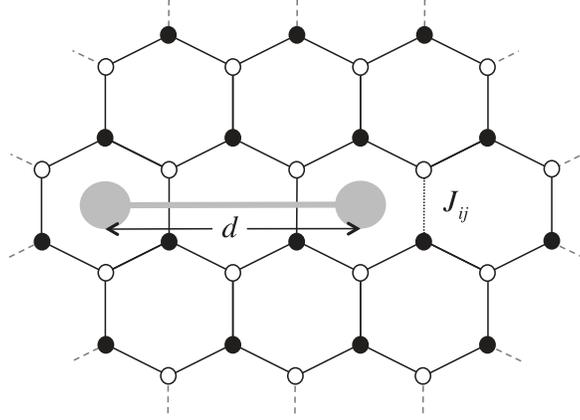


Figure 11: tuning  $J_z = -1$  on the  $d$  successive  $z$ -links that lie on links crossed by a string, which is equivalent to setting  $u_{ij} = -1$  on these links. This operation creates two vortices separated by distance  $d$ . Or in the continuous way, by setting  $J_{ij}(s) = 1 - 2s/S$  with  $s \in [0, S]$

In the absence of vortices, the fermionic spectrum has an energy gap  $\Delta$  (the zero-mode energy of a free fermion) above the ground state energy  $E_0^{0v}$ . The gap remains as well when a pair of vortices separated by  $d$  is introduced, whose ground state energy is to be denoted by  $E_d^{2v}$ , and excitation by  $E_d^{2v} + \epsilon^d$ . In Fig.(12), the ground state energy of 2-vortex sector  $E_d^{2v}$  and its first excited states  $E_d^{2v} + \epsilon^d$  collapse into the same 2-vortex energy  $\Delta_{2v}$  at large separation, i.e. the ground state becomes two-fold degenerate.

For convenience, let us define the creation operator of  $i$ -th excitation (or  $i - 1$  fermionic mode) to be  $z_i^\dagger$ , and the ground state of  $n$ -vortex sector to be  $|\Psi_0^{nv}\rangle$ . In the occupation basis, the states in 2-vortex sector can be written as:

$$|0\rangle = |\Psi_0^{2v}\rangle, \quad |1\rangle = z_1^\dagger |0\rangle \quad (3.16)$$

At large distance,  $|0\rangle$  and  $|1\rangle$  are two degenerate states with energy  $\Delta_{2v}$  above vortex-free ground state, thereby making the two states energetically indistinguishable. Their wavefunctions differ by the occupation of the gapless zero mode. However, for small distance, the zero mode acquires an energy gap above  $E_d^{2v}$ , thus the 2-fold degeneracy will be lifted, and the two states become energetically distinguishable. As  $d \rightarrow 0$ , the fusion of two vortices brings the ground state wavefunction back to vortex-free sector, the zero-mode is then gapped by the original gap  $\Delta$ .

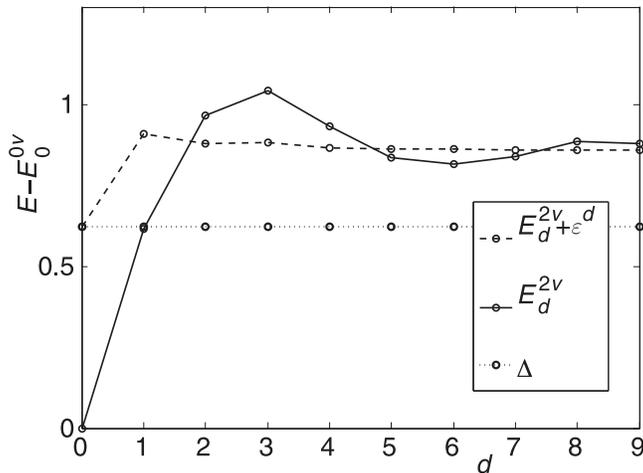


Figure 12: The gap behavior for a 2-vortex configuration as a function of the vortex separation  $d$ . The spectrum is plotted against the ground state energy of 0-vortex  $E_0^{0v}$ . The solid line is the total ground state energy in 2-vortex sector, denoted by  $E_d^{2v}$ . The dashed line is the energy of the first excited state in 2-vortex sector, which is  $\epsilon^d$  above the ground state. The dotted line  $\Delta$  is the lowest free-fermion energy in all-vortex sector.

### 3.3 Fusion Rule and Non-Abelian Statistics

In this section, I will brief sketch the fusion corresponds to bringing two anyons together and determines how they behave collectively. The detailed derivation is to be done out of this term paper project in the coming summer term. The distinct behavior of  $|0\rangle$  and  $|1\rangle$  discussed in last section at small  $d$  is indicative that the occupation of the fermionic zero mode corresponds to the fusion channel of the vortices. Therefore we identify these states with Ising anyons:

$$\begin{aligned} \text{Ground state} &\Leftrightarrow 1, \text{ vac.} \\ \text{Vortex excitation} &\Leftrightarrow \sigma, \text{ non-Abelian anyon} \\ \text{Fermionic excitation} &\Leftrightarrow \psi, \text{ fermion} \end{aligned} \quad (3.17)$$

The non-trivial fusion rules are given by:

$$\psi \times \psi = 1, \quad \psi \times \sigma = \sigma, \quad \sigma \times \sigma = 1 + \psi \quad (3.18)$$

To see the non-Abelian statistics of vortices, consider 4 vortices created in pairs. The behaviour of Ising anyons undergoing the described evolution is given

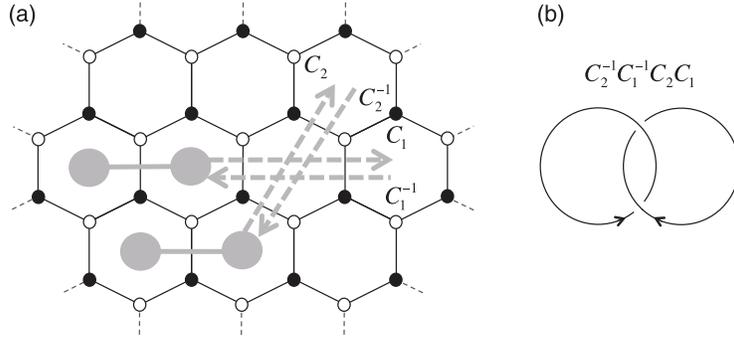


Figure 13: Two pairs of vortices are created in lattice. The paths in (a) is topologically equivalent to a link shown in (b)

in Fig.(13). Such a braiding operation is predicted to produce a change by:

$$B = e^{-\pi/4i} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (3.19)$$

Unlike in the toric code model where braiding operation produces  $U(1)$  phase factor, in the non-abelian phase the effect of braiding is a matrix.

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