Basics about Entanglement

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Theorem 1. *Let* $|\psi\rangle$ *be a pure quantum state of a lattice Hamiltonian. For an operator* O *that is supported on one part of the bipartited of lattice the following holds:*

$$\langle\psi|O|\psi\rangle = \mathrm{Tr}(\rho_i O) \tag{1}$$

where ρ_i *is the (reduced) density matrix of the selected part of lattice, and the trace is taken over either states thereof or of the full system (they give the same result).*

Proof. The full wavefunction can be decomposed into

$$|\psi\rangle = \sum_{i,o} \omega_{i,o} |\phi_i\rangle |\phi_o\rangle \tag{2}$$

and ρ_i is

$$\begin{aligned} \rho_i = \mathrm{Tr}_o(|\psi\rangle\langle\psi|) &= \sum_o \langle\phi_o|\psi\rangle\langle\psi|\phi_o\rangle \\ &= \sum_{i,i'} \left(\sum_o \omega_{i,o}\, \omega^*_{o,i'}\right) |\phi_i\rangle\langle\phi_{i'}| \end{aligned} \tag{3}$$

Measurement tracing density matrix over all $|\psi_i\rangle$ is

$$\begin{aligned} \mathrm{Tr}(\rho_i O) &= \sum_i \langle\phi_i|\rho_i O|\phi_i\rangle \\ &= \sum_{i,i'} \left(\sum_o \omega_{i,o}\, \omega^*_{o,i'}\right) \langle\phi_{i'}|O|\phi_i\rangle \end{aligned} \tag{4}$$

Note that one can also do a trace over states of the full lattice as long as $|\phi_o\rangle$ is normalized, which doesn't matter since O acting only on $|\phi_i\rangle$. On the other hand, a direct meaure by $\langle\psi|O|\psi\rangle$ is

$$\begin{aligned} \langle\psi|O|\psi\rangle &= \sum_{i',o'} \sum_{i,o} \omega^*_{o'i'}\omega_{i,o} \langle\phi_{o'}\phi_{i'}| O |\phi_i\phi_o\rangle \\ &= \sum_{i,i'} \left(\sum_o \omega_{i,o}\, \omega^*_{o,i'}\right) \langle\phi_{i'}|O|\phi_i\rangle \end{aligned} \tag{5}$$

where in the last contraction we used the fact that O is supported on part i of Hilbert space. □

Theorem 2. *If* ρ *and* ρ' *act on disjoint Hilbert spaces, then the von-Neumann entanglement entropy satisfies*

$$S(\rho\otimes\rho') = S(\rho) + S(\rho')$$

Proof. Since entanglement is invariant under change of basis, we can diagonalize ρ and ρ' and calculate EE using their diagonal elements:

$$\tilde{\rho} = \sum_i a_i \left|a_i\right\rangle \left\langle a_i\right| , \quad \tilde{\rho}' = \sum_j b_j \left|b_j\right\rangle \left\langle b_j\right|$$

so the compoiste matrix used in $S(\rho \otimes \rho')$ is

$$\tilde{\rho} \otimes \tilde{\rho}' = \sum_{ij} a_i b_j \left|a_i b_j\right\rangle \left\langle b_j a_i\right|$$

$$\log\left(\tilde{\rho} \otimes \tilde{\rho}'\right) = \sum_{ij} (\log a_i + \log b_j) \left|a_i b_j\right\rangle \left\langle b_j a_i\right|$$

the total entanglement entropy is thus

$$\begin{aligned} S(\tilde{\rho} \otimes \tilde{\rho}') &= - \sum_{ij} a_i b_j \log(a_i) + a_i b_j \log(b_i) \\ &= - \sum_i a_i \log(a_i) \sum_j b_j - \sum_j b_j \log(b_j) \sum_i a_i \\ &= - \sum_i a_i \log(a_i) - b_i \log(b_i) \\ &= S(\tilde{\rho}) + S(\tilde{\rho}') \end{aligned}$$

where we used $\mathrm{Tr}(\rho) = 1$ in the 2nd step. □

Theorem 3. *von-Neumann entanglement entropy is maximal iff there exists a basis in which its (reduced) density matrix is an equally weighted diagonal matrix.*

Proof. In a diagonalized reduced density matrix $diag(\epsilon_1, \ldots \epsilon_n)$, the von-Neumann entropy is

$$S = - \sum_i \epsilon_i \log \epsilon_i \quad S.t. \ \sum_i \epsilon_i = 1$$

To see its maximum apply Lagrangian multiplier

$$L(\vec{\epsilon}, \lambda) = \sum_i -\epsilon_i \log \epsilon_i - \lambda \left(\sum_i \epsilon_i - 1 \right)$$

by derivative wrt ϵ_j we have

$$\forall \epsilon_i, \quad \log \epsilon_i + 1 + \lambda = 0$$

there's only one constraint characterized by λ, so all ϵ_i must have the same value. □

Definition 0.1 (Maximally entangled state)**.** For a bipartite system $A \cup B$, the wavefunction of a maximally entangled state is given by

$$|\psi\rangle = \frac{1}{\sqrt{D}} \sum_{i=1}^{D} |i_A\rangle \otimes |i_B\rangle \tag{6}$$

with $D = \min(\dim \mathcal{H}_A, \dim \mathcal{H}_B)$, $\langle i | i' \rangle = \delta_{i,i'}$.

The interpretation as maximally entangled comes from the fact that

$$\rho_A = \frac{1}{D_A}\mathbb{1}_A$$

where we assumed $D = D_A$ WLOG. As a result, we must have

$$S(\rho_A) = \sum_i \frac{1}{D_A} \log D_A = \log D_A \sim \text{vol } A$$

where $\log D_A \sim \text{vol } A$ is because (assuming 1 qubit per site)

$$\log D_A = \log\left(2^{\text{number of sites in A}}\right) \propto \text{number of sites in A}$$

which is proptional to the volumn of A in a system where dofs are uniformly distributed. This is the famous bound on EE:

$$0 \leq S(\rho_A) \leq \log D_A \propto \text{vol } A \tag{7}$$